CCA Math Bonanza
March 8, 2025
Division II Solutions

Contents

CCA Math Bonanza

March 8, 2025

Division II Individual Round

I1) Rupert the Raven just won the lottery in CCA land! However, the government will take 20% of his winnings as tax. After paying taxes, Rupert wants to have exactly 1 thousand dollars. Find the amount of money Rupert needs to have won.

I2) When Lukas turned 7 years old, his mother told him that she was 5 times as old as him. Find the age of his mother when Lukas turns 60 years old.

I3) Find the value of

$$\sqrt{(2-0-2-5)(2+0+2+5)(2-0+2-5)(2+0-2+5)}.$$

I4) Find the number of points (x, y) on the coordinate plane such that x and y are positive integers and $x + y \leq 7$.

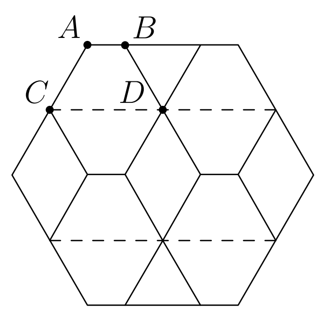
I5) Let $a \heartsuit b$ be the result of dividing a by b and keeping only the integer part. For example, $11 \heartsuit 3 = 3$ because $\frac{11}{3} = 3\,\frac{2}{3}$, and $29 \heartsuit 7 = 4$ because $\frac{29}{7} = 4\,\frac{1}{7}$. Find $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$.

I6) Byan picks a random integer between 0 and 4, inclusive. Then, Rai flips 4 fair coins. The probability the amount of heads Rai lands is equal to Byan's integer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

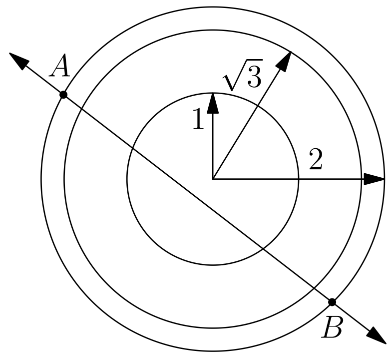
I7) The sum of the coefficients (including the constant term) of the polynomial $(x^2 + ax - 4)^2$ is 36. Find the sum of the squares of all possible values of a.

I8) Let y be an integer, and let $x = 5^4 - 5^2y^2$. Given that x is an even positive integer, find the largest possible number of positive divisors of x.

I9) The diagram below shows eight congruent isosceles trapezoids arranged inside a regular hexagon. $\frac{AB}{CD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.



I10) In the diagram below, there are three circles of radii 1, $\sqrt{3}$, and 2, all with the same center. Points A and B are chosen uniformly at random along the circumference of the outermost circle. In the diagram, the line AB intersects the circle six times. The probability that line AB intersects the circles exactly four times can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.



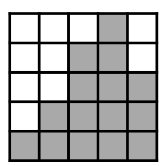
I11) Let $f(n)$ denote the number of 1s in the base 2 representation of n. For example, $f(5) = 2$. Let $s(n) = f(1) + f(2) + \cdots + f(n)$. Find $s(2^0) + s(2^1) + \cdots + s(2^9)$.

I12) Let P and Q be fixed points such that $PQ = 12$. Let A and B be points such that P lies on AB and $AQ : QB : BA = 3 : 4 : 5$. Find the maximum area of $\triangle AQB$ over all valid locations of A and B.

I13) An n-tower is constructed by shading cells in an $n \times n$ grid according to the following rules:

- The kth row from the top has k consecutive shaded cells.
- Each shaded cell not in the bottom row is directly above another shaded cell.

An example of a 5-tower is shown below.



Let A be the number of 25-towers such that the middle cell of the 1st row from the top and the leftmost cell of the 21st row from the top are shaded. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

I14) Let ℓ be a non-vertical line that does not intersect the curve $y = \frac{x^4 - x^3}{x - 1}$. Let A be the smallest real number which is larger than all possible slopes of such a line ℓ. A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I15) Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 5$, and $CA = 4$. Let D and E be points on segments BC and AC, respectively. The minimum value of $AD + DE + EB$ over all choices of D and E can be expressed as $\frac{a\sqrt{b}}{c}$, where b is a positive integer not divisible by the square of any prime, and a and c are relatively prime positive integers. Find $a + b + c$.

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Division II Team Round

T1) Byan Rai is deciding what to wear to school today. He has three shirts that are blue, red, and yellow, and he has four pairs of pants that are blue, yellow, green, and orange. He will pick one shirt and one pair of pants to wear, but he doesn't want both of them to be the same color. Find the number of ways for him to dress.

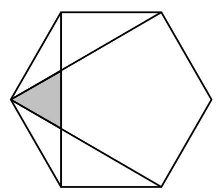
T2) The value

$$\frac{25!^2 - 24!^2}{25!^2 + 24!^2}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

T3) Find the number of primes less than 100 which can be written in the form $a^4 + b^4$, where a and b are (not necessarily distinct) positive integers.

T4) The regular hexagon shown in the figure below has side length 1. The area of the shaded region can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.



T5) Alex picks a positive integer a, and Bryan picks a positive integer b. Alex's number has 2 digits both when written in base 12 and when written in base b. Find the sum of all possible values of b.

T6) Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let D be the foot of the altitude from A to BC. Let P be the point on segment AC such that $AP = AD$. Let Q be the point on segment AD such that PQ and BC are parallel. Find the area of $\triangle AQB$.

T7) Six boys and thirteen girls sit at a circular table. Over all possible ways for them to be seated, find the minimum possible number of people that have a girl to the left and right of them.

T8) Freddy the frog is currently standing at point $(1, 1)$ on the coordinate plane. Every minute, Freddy picks a cardinal direction (north, south, east, or west) uniformly at random and hops either 1 or 2 units, picking his hop length uniformly at random as well. He stops after hopping over either $x = 0.5, x = 2.5, y = 0.5$, or $y = 2.5$. The probability he hops over $x = 0.5$ or $y = 0.5$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

T9) Find the number of positive integers n such that $n \leq 750$ and $\gcd(1500, n)$ is a prime number.

T10) Let d be a real number, and let a, b, and c be the roots of the polynomial $x^3 - dx^2 + 8x + 2$. Given that

$$\frac{ab}{a+b} + \frac{ac}{a+c} + \frac{bc}{b+c} = 2,$$

the value of d can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

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Division II Lightning Round

Set 1

Each problem in this set is worth 10 points.

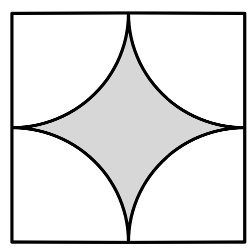
L1.1) Find the number of points (x, y) on the coordinate plane such that x and y are positive integers that are at most 9, and the units digit of xy is 7.

L1.2) Three numbers have a max of 27, a min of 13, and a mean of 21. Find their median.

L1.3) Find the remainder when the following expression is divided by 2025.

$$2^{0^{2^{5^{2^{0^{2^{5^{\cdot^{\cdot^{\cdot}}}}}}}}}}$$

L1.4) The diagram below contains 4 congruent quarter-circles. The side length of the outer square is 2. The perimeter of the shaded region can be expressed as $n\pi$, where n is a positive integer. Find n.

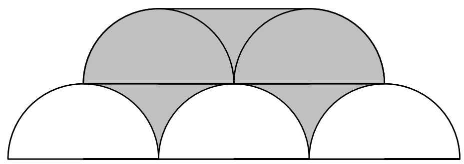


Set 2

Each problem in this set is worth 12 points.

L2.1) Rupert the Raven and Freddy the Falcon are fighting over the last banana. They will each roll a fair n-sided die ($n > 4$), whose faces are labeled with the integers from 1 to n. If the larger of the two rolls is at most 4, Freddy wins. Otherwise, Rupert wins. Find the value of n such that Rupert is 15 times as likely to win as Freddy.

L2.2) The diagram below contains 5 congruent semicircles of radius 1. The shaded region can be expressed as $a - \frac{b}{c}\pi$, where a, b, and c are positive integers and b and c are relatively prime. Find $a + b + c$.



L2.3) Find the sum of the digits of

$$-20 \times -19 \times -18 \times \cdots \times 23 \times 24 \times 25.$$

L2.4) Aidan, Bryan, Charles, and Dylan walk into a movie theater and randomly sit down in a row of 4 seats, with all seating arrangements being equally likely. Unfortunately, Bryan hates both Aidan and Charles. The probability Bryan does not sit next to Aidan or Charles can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Set 3

Each problem in this set is worth 14 points.

L3.1) Alice and Bob are workers that build walls at a uniform rate. Alice can build a certain wall in 3 hours. Alice and Bob together can build that same wall in 1 hour and 36 minutes. Find the number of hours it will take Bob to build a wall that is 14 times as wide and 15 times as tall as the original wall.

L3.2) Zhenghua writes the numbers $1, 2, 3, ..., 2025$ on a whiteboard. Zhenghua wants to select one or more of them such that, when Zhenghua multiplies his selected numbers together, their product is divisible by 5. The number of ways Zhenghua can do this can be expressed as $2^m - 2^n$, where m and n are positive integers. Find $100m + n$.

L3.3) Let $ABCD$ be a rectangle. Let $CEFG$ be a square such that E lies on segment AB and F lies on segment CD. Given that $BE = DF$ and $DF + EF = 3$, the area of $ABCD$ can be expressed as $m - n\sqrt{2}$, where m and n are positive integers. Find $100m + n$.

L3.4) Byan and Rai each pick a positive integer less than 100. It turns out that Byan's number is divisible by 5 and is greater than Rai's number. In addition, Rai's number divides Byan's number. Find the sum of all possible numbers Rai could've picked.

Set 4

Each problem in this set is worth 16 points.

L4.1) Let $P(x)$ be a polynomial of degree 2025, and let $Q(x)$ be a polynomial of degree 2. The graph of $Q(x)$ is tangent to the graph of $P(x)$ at at least 20×25 different points. Find the maximum possible number of total intersections (counting the tangencies) the graph of $P(x)$ could have with the graph of $Q(x)$.

L4.2) Find the sum of all positive integers whose positive integer divisors (including itself) sum to 434.

L4.3) Rupert the Raven is playing a game. His score starts at 0. Every second, he flips a coin. If the coin lands heads, his score increases by 1. Otherwise, his score will reset to 0. Rupert repeats this for 100 coin flips. The expected number of times his score becomes 5 after a coin flip can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L4.4) A right circular cone containing an infinite series of spheres $S_0, S_1, S_2, \ldots$ has a base of radius 3 and a height of 4. Let S_0 be the largest sphere that can be placed entirely within the cone, and for $n > 0$, let S_n be the largest sphere that can be placed entirely within the space between the tip of the cone and S_{n-1}. The total volume of all the spheres can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

Note: $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$ and $\lfloor 14 \rfloor = 14$.

L5.1) The mex, or minimum excluded value, of a list of nonnegative integers is defined as the smallest nonnegative integer that does not appear in the list. For example, the mex of $0, 9, 2, 1, 1, 0$ is 3, because 0, 1, and 2 appear, but 3 does not.

Submit a nonnegative integer X. If M is the mex of all submissions to this problem, you will receive $\left\lfloor \frac{20}{(M-X)^2} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

L5.2) The sequence of Lucas numbers is defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Estimate the number of integers between 1 and 2025, inclusive, that can be expressed as the sum of 3 (not necessarily distinct) Lucas numbers.

Submit a positive integer E. If the correct answer is A, you will receive $\lfloor 20 \cdot 0.98^{|A-E|} \rfloor$ points.

L5.3) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Consider writing down $F_0, F_1, \ldots, F_{10000}$ and joining them all together. Estimate the number of times the digits $1, 2, 3$ occur consecutively in any order. For example, joining $F_0, F_1, \ldots, F_{10}$ gives 11235813213455, which contains 4 such occurrences.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot \min\left(\frac{A}{E}, \frac{E}{A}\right)^{13} \right\rfloor$ points.

L5.4) Submit an integer between 1 and 240, inclusive. Your *index* is initially 0. Every positive integer divisor x of your submission will add $\frac{1}{n}$ points to your index, where n is the number of submissions to this problem (including yours) that are divisible by x.

If your index is S and the maximum index amongst all teams is M, you will receive $\left\lfloor 20 \cdot \left(\frac{S}{M}\right)^{1.5} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

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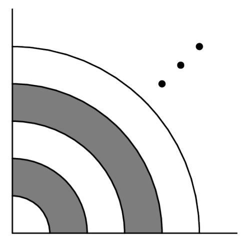
Division II Tiebreaker Round

TB1) Find the number of factors of 2025 that are multiples of 5 but not 27.

TB2) Shown below are 100 quarter circles of radii $1, 2, 3, \ldots, 100$, all of which share the same center and are oriented in the same direction. These circles divide the figure into 100 rings. The first ring is colored white, and subsequent rings alternate between grey and white. The area of the grey region can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.



TB3) Let n and x be positive integers such that the sum of the digits of x is 2 and

$$9900\,9900\,9900\,9901 \times n = x.$$

Find the smallest such n.

TB4) Charlotte the cat lives in Cartesia, a city on the coordinate plane whose roads are the lines $x = a$ and $y = a$ for integers a. Charlotte is currently standing at the origin, and would like to walk to the highway at $y = 5$, and then to her home at $(6, 1)$. Let D be the length of the shortest possible path she could take home. Find the distinct number of paths of length D Charlotte can take.

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Division II Individual Round Answer Key

I1) 1250

I2) 88

I3) 15

I4) 21

I5) 2

I6) 105

I7) 90

I8) 24

I9) 103

I10) 103

I11) 4107

I12) 150

I13) 59

I14) 304

I15) 31

CCA Math Bonanza

March 8, 2025

Division II Team Round Answer Key

T1) 10

T2) 31513

T3) 3

T4) 148

T5) 10290

T6) 24

T7) 7

T8) 508

T9) 240

T10) 1003

CCA Math Bonanza

March 8, 2025

Division II Lightning Round Answer Key

Set 1

Each problem in this set is worth 10 points.

L1.1) 4

L1.2) 23

L1.3) 1

L1.4) 2

Set 2

Each problem in this set is worth 12 points.

L2.1) 16

L2.2) 9

L2.3) 0

L2.4) 106

Set 3

Each problem in this set is worth 14 points.

L3.1) 720

L3.2) 204120

L3.3) 8154

L3.4) 385

Set 4

Each problem in this set is worth 16 points.

L4.1) 1525

L4.2) 1210

L4.3) 9764

L4.4) 3207

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

L5.1) The mex of all submissions was 7.

L5.2) 469

L5.3) 62571

L5.4) Here are all the valid submissions we received:
2 6 7 24 24 26 40 69 77 120 120 128 144 147 154 180 192 200 209 209 210 210 210 210 210 212 212 215 221 224 230 234 240 240 240 240 240 240 240 240 240 240

CCA Math Bonanza

March 8, 2025

Division II Tiebreaker Round Answer Key

TB1) 6

TB2) 252502

TB3) 101

TB4) 5005

CCA Math Bonanza

March 8, 2025

Division II Individual Round Solutions

I1) Rupert the Raven just won the lottery in CCA land! However, the government will take 20% of his winnings as tax. After paying taxes, Rupert wants to have exactly 1 thousand dollars. Find the amount of money Rupert needs to have won.

Proposed by Jason Yang.

Answer: 1250

Solution: After tax is applied, only 80% of the given money remains. Thus, our answer is $\frac{1000}{80\%} = 1250$.

I2) When Lukas turned 7 years old, his mother told him that she was 5 times as old as him. Find the age of his mother when Lukas turns 60 years old.

Proposed by Rohan Mallick.

Answer: 88

Solution: When Lukas was 7, his mother was 35 and thus $35 - 7 = 28$ years older than him. When Lukas is 60, his mother will then be $60 + 28 = 88$ years old.

I3) Find the value of

$$\sqrt{(2-0-2-5)(2+0+2+5)(2-0+2-5)(2+0-2+5)}.$$

Proposed by Zhenghua Xie.

Answer: 15

Solution: The above expression evaluates to

$$\sqrt{(-5)(9)(-1)(5)} = 15.$$

I4) Find the number of points (x, y) on the coordinate plane such that x and y are positive integers and $x + y \leq 7$.

Proposed by Ryan Bai.

Answer: 21

Solution: When $x = 1$, y can take on 6 possible values (any integer between 1 and 6, inclusive). When $x = 2$, y can take on 5 possible values. When $x = 3$, y can take on 4 possible values. This pattern continues until $x = 6$, for which y can take on 1 possible value. Our answer is then

$$1 + 2 + 3 + \cdots + 6 = 21.$$

I5) Let $a \heartsuit b$ be the result of dividing a by b and keeping only the integer part. For example, $11 \heartsuit 3 = 3$ because $\frac{11}{3} = 3\,\frac{2}{3}$, and $29 \heartsuit 7 = 4$ because $\frac{29}{7} = 4\,\frac{1}{7}$. Find $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$.

Proposed by Jason Yang.

Answer: 2

Solution: We wish to compute

$$\frac{\frac{2025}{3}}{\frac{2024}{7}},$$

but keeping only the integer part of each division. Since 2025 and 2024 are so close to each other, we can simplify $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$ to $7 \heartsuit 3 = 2$.

I6) Byan picks a random integer between 0 and 4, inclusive. Then, Rai flips 4 fair coins. The probability the amount of heads Rai lands is equal to Byan's integer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ryan Bai.

Answer: $105\ \left(\frac{1}{5}\right)$

Solution: Imagine if Byan picked his random integer after Rai flipped his coins, since these two events are independent. The probability Byan picks the right number of heads is just $\frac{1}{5}$.

I7) The sum of the coefficients (including the constant term) of the polynomial $\left(x^2 + ax - 4\right)^2$ is 36. Find the sum of the squares of all possible values of a.

Proposed by Aditya Bisain.

Answer: 90 ($a = 9$ or $a = -3$)

Solution: Let $P(x) = \left(x^2 + ax - 4\right)^2$. The sum of the coefficients of P is $P(1)$. Thus, we have

$$P(1) = (a - 3)^2 = 36,$$

so $a - 3 = \pm 6$, yielding solutions of $a = 9$ and $a = -3$.

I8) Let y be an integer, and let $x = 5^4 - 5^2 y^2$. Given that x is an even positive integer, find the largest possible number of positive divisors of x.

Proposed by Aditya Bisain.

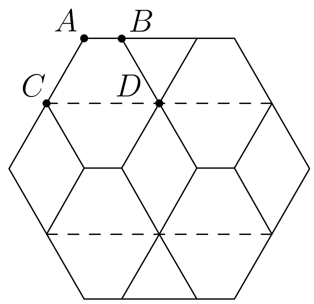
Answer: 24

Solution: We have $x = 5^2\left(5^2 - y^2\right)$. Since x is positive, we have $|y| < 5$. Furthermore, x is even, so y must be odd. Thus, $|y|$ is either 1 or 3.

- When $|y| = 1$, $x = 25 \times 24 = 2^3 \times 3 \times 5^2$ which has $4 \times 2 \times 3 = 24$ factors.
- When $|y| = 3$, $x = 25 \times 16 = 2^4 \times 5^2$ which has $5 \times 3 = 15$ factors.

Taking the larger of the two gives our final answer of 24.

I9) The diagram below shows eight congruent isosceles trapezoids arranged inside a regular hexagon. $\frac{AB}{CD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m+n$.

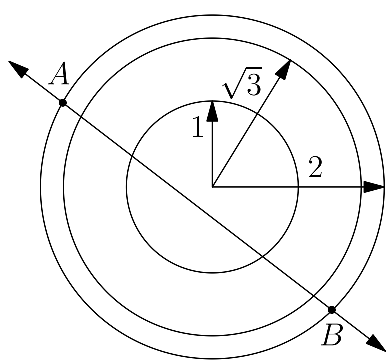


Proposed by Jason Yang.

Answer: $103\left(\frac{1}{3}\right)$

Solution: Without loss of generality, let the side length of the hexagon be 4. Then, $AC = BD = \frac{4}{2} = 2$. Let E be the foot of the altitude from A to CD and F be the foot of the altitude from B to CD. Since ACE is a 30-60-90 triangle, $CE = \frac{1}{2}AC = 1$, and $FD = \frac{1}{2}BD = 1$. Thus, $CD = CE + EF + FD = 1 + 1 + 1 = 3$. Also, the smaller triangle at the top of the hexagon is an equilateral triangle of side length 2, so $AB = 1$. Our answer is then $\frac{AB}{CD} = \frac{1}{3}$.

I10) In the diagram below, there are three circles of radii 1, $\sqrt{3}$, and 2, all with the same center. Points A and B are chosen uniformly at random along the circumference of the outermost circle. In the diagram, the line AB intersects the circle six times. The probability that line AB intersects the circles exactly four times can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m+n$.



Proposed by Jason Yang.

Answer: $103\left(\frac{1}{3}\right)$

Solution: Let ω_1, ω_2, and ω_3 be the three circles, with ω_1 referring to the smallest circle and ω_3 the largest. Without loss of generality, assume A is placed at the leftmost point of ω_3 and that B is placed in the top half of ω_3. Now, the possible places we could place B is an arc on ω_3 bounded by the tangent lines from A to ω_1 and ω_2, since line AB intersects exactly ω_2 and ω_3.

By the tangency, two right triangles can be formed to determine the angle subtended by the arc. The inscribed angle of the arc is $60° - 30° = 30°$, so its central angle is $60°$. Dividing by $180°$ (since we assume B is only in the top half of the circle) gives a final answer of $\frac{1}{3}$.

I11) Let $f(n)$ denote the number of 1s in the base 2 representation of n. For example, $f(5) = 2$. Let $s(n) = f(1) + f(2) + \cdots + f(n)$. Find $s(2^0) + s(2^1) + \cdots + s(2^9)$.

Proposed by Andrew Jahng.

Answer: 4107

Solution: Consider

$$s(2^k - 1) = f(0) + f(1) + f(2) + \cdots + f(2^k - 1).$$

For each of the first k binary digits, it is a 1 in exactly 2^{k-1} of the numbers between 0 and $2^k - 1$. Thus,

$$s(2^k - 1) = k \times 2^{k-1}.$$

Our original expression is then

$$\begin{aligned} & s(2^0) + s(2^1) + \cdots + s(2^9) \\ &= (s(2^0 - 1) + f(2^0)) + (s(2^1 - 1) + f(2^1)) + \cdots + (s(2^9 - 1) + f(2^9)) \\ &= (0 + 1) + (1 \times 2^0 + 1) + (2 \times 2^1 + 1) + \cdots + (9 \times 2^8 + 1) \\ &= 10 + 1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \cdots + 9 \times 2^8. \end{aligned}$$

Let

$$S = 1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 = \cdots + 9 \times 2^8.$$

S is the sum of an arithmetico-geometric series. To evaluate S, note that

$$2S = 0 \times 2^0 + 1 \times 2^1 + 2 \times 2^2 + \cdots + 8 \times 2^8 + 9 \times 2^9,$$

so

$$\begin{aligned} 2S - S &= -2^0 - 2^1 - 2^2 - \cdots - 2^8 + 9 \times 2^9, \\ &= -(2^9 - 1) + 9 \times 2^9 \\ &= 4097. \end{aligned}$$

Our answer is then $S + 10 = 4107$.

I12) Let P and Q be fixed points such that $PQ = 12$. Let A and B be points such that P lies on AB and $AQ : QB : BA = 3 : 4 : 5$. Find the maximum area of $\triangle AQB$ over all valid locations of A and B.

Proposed by Rohan Mallick.

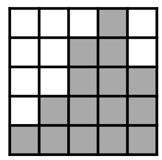
Answer: 150

Solution: Since we want to make the triangle as large as possible, we want the fixed length PQ to be as small as possible relative to the triangle. Therefore, we make PQ the Q-altitude of the triangle. By the ratio statement, $\triangle AQB$ has a right angle at Q, so we can use similar triangles to get that $AQ = 15$ and $QB = 20$, giving an area of 150.

I13) An n-tower is constructed by shading cells in an $n \times n$ grid according to the following rules:

- The kth row from the top has k consecutive shaded cells.
- Each shaded cell not in the bottom row is directly above another shaded cell.

An example of a 5-tower is shown below.



Let A be the number of 25-towers such that the middle cell of the 1st row from the top and the leftmost cell of the 21st row from the top are shaded. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

Proposed by Zhenghua Xie.

Answer: 59 $\left(\binom{20}{8} = 2 \times 3 \times 5 \times 13 \times 17 \times 19\right)$

Solution: Consider constructing the tower from the top down, going row by row. At each row, after shading in all squares present in the row above us, we can choose to extend the consecutive shaded cells either one cell to the left or one cell to the right.

Call a row a leftie if we extended to the left and call it a rightie if we extended one cell to the right. The middle cell of the first row being shaded tells us that, of the 24 other rows, 12 are lefties and 12 are righties. The leftmost cell of the 21st row being shaded tells us that the last 4 rows are all righties. Now, we need to choose 12 of the 20 rows in the middle to be lefties, so our answer is

$$\binom{20}{8} = 2 \times 3 \times 5 \times 13 \times 17 \times 19.$$

I14) Let ℓ be a non-vertical line that does not intersect the curve $y = \frac{x^4 - x^3}{x-1}$. Let A be the smallest real number which is larger than all possible slopes of such a line ℓ. A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ryan Wu.

Answer: 304 $\left(\frac{3}{4}\right)$

Solution: The graph of $y = \frac{x^4 - x^3}{x-1}$ is just $y = x^3$ but with a hole at $(1, 1)$. We need ℓ to pass through this hole. Let the equation of ℓ be $y = m(x-1) + 1$. ℓ should intersect $y = x^3$ twice: once at $(1, 1)$ and another at a point of tangency. Let's compute the x-coordinate of all intersections between ℓ and $y = x^3$.

Substitute and rearrange to get

$$x^3 - mx + (m-1) = 0.$$

We know $x = 1$ is a solution, so factoring out $(x-1)$ from the left hand side gives

$$x^2 + x + (1-m) = 0.$$

We want this quadratic to have exactly one solution, so we set its discriminant to 0 and solve to find that the maximum value of the slope is $\frac{3}{4}$.

I15) Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 5$, and $CA = 4$. Let D and E be points on segments BC and AC, respectively. The minimum value of $AD + DE + EB$ over all choices of D and E can be expressed as $\frac{a\sqrt{b}}{c}$, where b is a positive integer not divisible by the square of any prime, and a and c are relatively prime positive integers. Find $a + b + c$.

Proposed by Aidan Bai.

Answer: $31\ \left(\frac{9\sqrt{17}}{5}\right)$

Solution: Let A' be the reflection of A across line BC, and let B' be the reflection of B across line CA. Note that by the reflection, $AD = A'D$ and $BE = B'E$, so $AD + DE + EB = A'D + DE + EB'$. Since the shortest path between two points is the line connecting the two, we have

$$AD + DE + EB = A'D + DE + EB' \geq A'B'.$$

We can in fact achieve this minimum by letting D and E be the points where line $A'B'$ intersects lines CB and CA, respectively.

We use the Pythagorean Theorem to finish. Let E be the foot of the altitude from A' to line AB. Note that $\angle A'AE = 90^\circ - \angle A'AC = \angle ACB$, so $\triangle A'AE$ is a 3-4-5 triangle with hypotenuse $\frac{24}{5}$. By similar triangles, we can compute $AE = \frac{96}{25}$ and $EA' = \frac{72}{25}$, and $B'E = B'A + AE = 3 + \frac{96}{25} = \frac{171}{25}$. Then,

$$\begin{aligned}
B'A' &= \sqrt{B'E^2 + EA'^2} \\
&= \frac{1}{25}\sqrt{171^2 + 72^2} \\
&= \frac{9}{25}\sqrt{19^2 + 8^2} \\
&= \frac{9}{25}\sqrt{425} \\
&= \frac{9\sqrt{17}}{5}.
\end{aligned}$$

CCA Math Bonanza

March 8, 2025

Division II Team Round Solutions

T1) Byan Rai is deciding what to wear to school today. He has three shirts that are blue, red, and yellow, and he has four pairs of pants that are blue, yellow, green, and orange. He will pick one shirt and one pair of pants to wear, but he doesn't want both of them to be the same color. Find the number of ways for him to dress.

Proposed by Ryan Bai.

Answer: 10

Solution: We can complementary count. Byan has $3 \times 4 = 12$ total ways of deciding what to wear today. Our of those, two of the combinations have him wearing the same color of shirt and pants, so our answer is $12 - 2 = 10$.

T2) The value

$$\frac{25!^2 - 24!^2}{25!^2 + 24!^2}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Jason Yang.

Answer: 31513 $\left(\frac{312}{313}\right)$

Solution:

$$\frac{25!^2 - 24!^2}{25!^2 + 24!^2} = \frac{(24!)^2(25^2 - 1^2)}{(24!)^2(25^2 + 1^2)} = \frac{624}{626} = \frac{312}{313}$$

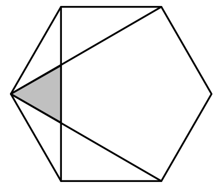
T3) Find the number of primes less than 100 which can be written in the form $a^4 + b^4$, where a and b are (not necessarily distinct) positive integers.

Proposed by Aidan Bai.

Answer: 3

Solution: Note that we must have $a \leq 3$ and $b \leq 3$, since $4^4 > 100$. Enumerating all such a and b yields $2 = 1^4 + 1^4$, $17 = 1^4 + 2^4$, and $97 = 2^4 + 3^4$ for a total of 3 such primes.

T4) The regular hexagon shown in the figure below has side length 1. The area of the shaded region can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.



Proposed by Jason Yang.

Answer: 148 $\left(\sqrt{\frac{1}{48}}\right)$

Solution: The shaded region is an equilateral triangle with a side length that is $\frac{1}{3}$ the length of the line segment across the hexagon. In fact, its side length is $\frac{\sqrt{3}}{3}$. The area of the equilateral triangle is then

$$\frac{\sqrt{3}}{4} * \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\sqrt{3}}{12} = \sqrt{\frac{1}{48}}$$

T5) Alex picks a positive integer a, and Bryan picks a positive integer b. Alex's number has 2 digits both when written in base 12 and when written in base b. Find the sum of all possible values of b.

Proposed by Zhenghua Xie.

Answer: 10290

Solution: Alex's integer has to be between 12 and 143, inclusive, for it to have 2 digits when written in base 12, and it has to be between b and $b^2 - 1$, inclusive, for it to have 2 digits when written in base b.

Thus, we need the intervals $[12, 143]$ and $[b, b^2 - 1]$ to have a nonempty intersection. This occurs when $4 \leq b \leq 143$, for an answer of $4 + 5 + \cdots + 143 = 10290$.

T6) Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let D be the foot of the altitude from A to BC. Let P be the point on segment AC such that $AP = AD$. Let Q be the point on segment AD such that PQ and BC are parallel. Find the area of $\triangle AQB$.

Proposed by Aditya Bisain.

Answer: 24

Solution: $AP = AD = 12$, so $\triangle AQP \sim \triangle ADC$ with ratio $12 : 15$. Thus, $AQ = AD \times \frac{12}{15} = \frac{48}{5}$. The area of $\triangle AQB$ is then $\frac{1}{2} AQ \cdot DB = 24$.

T7) Six boys and thirteen girls sit at a circular table. Over all possible ways for them to be seated, find the minimum possible number of people that have a girl to the left and right of them.

Proposed by Aidan Bai.

Answer: 7

Solution: If someone is sitting next to two girls, that means they are sitting next to no boys. Thus, the problem is equivalent to maximizing the number of people that are sitting next to at least one boy.

There can be at most $2 \times 6 = 12$ such people, since each boy only sits next to two people. The minimum possible number of people next to a girl is thus at least $19 - 12 = 7$. We can achieve this minimum with the following construction: `BGGBGGBGGBGGBGGBGGG`.

T8) Freddy the frog is currently standing at point $(1, 1)$ on the coordinate plane. Every minute, Freddy picks a cardinal direction (north, south, east, or west) uniformly at random and hops either 1 or 2 units, picking his hop length uniformly at random as well. He stops after hopping over either $x = 0.5, x = 2.5, y = 0.5$, or $y = 2.5$. The probability he hops over $x = 0.5$ or $y = 0.5$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Tiger Han.

Answer: 508 $\left(\frac{5}{8}\right)$

Solution: We can split this into cases.

Case 1: Freddy hops west or south. Freddy succeeds with probability 1 in this case. This case happens with probability $\frac{1}{2}$

Case 2.1: Freddy hops north two units. Freddy fails in this case, with this case occurring with probability $\frac{1}{8}$

Case 2.2: Freddy hops north one unit. By symmetry, Freddy is equally likely to succeed or fail, so he succeeds with probability $\frac{1}{2}$. This case occurs with probability $\frac{1}{8}$

Freddy hopping east one or two units is equivalent to Cases 2.1 and 2.2.

Combining these cases, we get that the chance of success is

$P = \frac{1}{2}(1) + \frac{1}{8}(0) + \frac{1}{8}\left(\frac{1}{2}\right) + \frac{1}{8}(0) + \frac{1}{8}\left(\frac{1}{2}\right) = \frac{5}{8}$.

T9) Find the number of positive integers n such that $n \leq 750$ and $\gcd(1500, n)$ is a prime number.

Proposed by Andrew Jahng.

Answer: 240

Solution: If $\gcd(1500, n)$ is a prime number, then $\gcd(1500, 1500 - n) = \gcd(1500, n)$ is also a prime number. Furthermore, $n = 750$ doesn't work. Thus, it suffices to count the number of $n \leq 1500$ for which $\gcd(1500, n)$ is prime and divide by 2.

Let $\phi(n)$, Euler's totient function, be the number of positive integer $x \leq n$ for which $\gcd(n, x) = 1$. Let's count the number of $n \leq 1500$ for which $\gcd(1500, n) = 3$. Let $n = 3k$, with $k \leq 500$. We want $\gcd(1500, 3k) = 3$, so $\gcd(500, k) = 1$. The number of such k is given by $\phi(500)$. Repeating this for the other prime factors of 1500 gives an answer of

$$\phi\left(\frac{1500}{2}\right) + \phi\left(\frac{1500}{3}\right) + \phi\left(\frac{1500}{5}\right) = 200 + 200 + 80 = 480.$$

Dividing by 2 gives a final answer of 240.

T10) Let d be a real number, and let a, b, and c be the roots of the polynomial $x^3 - dx^2 + 8x + 2$. Given that

$$\frac{ab}{a+b} + \frac{ac}{a+c} + \frac{bc}{b+c} = 2,$$

the value of d can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Sepehr Golsefidy.

Answer: 1003 $\left(\frac{10}{3}\right)$

Solution: The sum can be written as

$$\frac{1}{\left(\frac{1}{a}\right) + \left(\frac{1}{b}\right)} + \frac{1}{\left(\frac{1}{a}\right) + \left(\frac{1}{c}\right)} + \frac{1}{\left(\frac{1}{b}\right) + \left(\frac{1}{c}\right)} = 2.$$

Substituting $u = \frac{1}{a}$, $v = \frac{1}{b}$, and $w = \frac{1}{c}$ gives

$$\frac{1}{u+v} + \frac{1}{u+w} + \frac{1}{v+w} = 2,$$

and u, v, w are roots of $f(x) = 2x^3 + 8x^2 - dx + 1$. Adding the fractions of the left hand side gives

$$\frac{u^2 + v^2 + w^2 + 3uv + 3uw + 3vw}{(u+v)(u+w)(v+w)} = 2.$$

Note that, by Vieta's, $u + v = -4 - w$ and symmetric variants, so the denominator of the left hand side is $(-4-w)(-4-v)(-4-u) = \frac{f(-4)}{2} = \frac{4d+1}{2}$. The numerator is equal to $(-4)^2 - \frac{d}{2}$. Thus, we have

$$\frac{16 - \frac{d}{2}}{\frac{4d+1}{2}} = 2.$$

Solving gives $d = \frac{10}{3}$.

CCA Math Bonanza
March 8, 2025

Division II Lightning Round Solutions

Set 1

Each problem in this set is worth 10 points.

L1.1) Find the number of points (x, y) on the coordinate plane such that x and y are positive integers that are at most 9, and the units digit of xy is 7.

Proposed by Zhenghua Xie.

Answer: 4

Solution: We can directly enumerate all such pairs. When $x \neq y$, we get $(1, 7)$, $(7, 1)$, $(3, 9)$ and $(9, 3)$. No pairs with $x = y$ work.

L1.2) Three numbers have a max of 27, a min of 13, and a mean of 21. Find their median.

Proposed by Ryan Bai.

Answer: 23

Solution: The numbers sum to $3 \cdot 21 = 63$, so their median is $63 - 13 - 27 = 23$.

L1.3) Find the remainder when the following expression is divided by 2025.

$$2^{0^{2^{5^{2^{0^{2^{5^{\cdot^{\cdot^{\cdot}}}}}}}}}}$$

Proposed by Andrew Jahng.

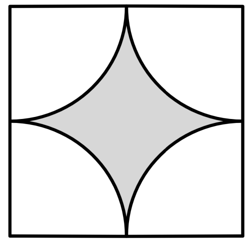
Answer: 1

Solution: Note that, by order of operations, we first evaluate

$$0^{2^{5^{2^{0^{2^{5^{\cdot^{\cdot^{\cdot}}}}}}}}} = 0,$$

then evaluate $2^0 = 1$ for our final answer.

L1.4) The diagram below contains 4 congruent quarter-circles. The side length of the outer square is 2. The perimeter of the shaded region can be expressed as $n\pi$, where n is a positive integer. Find n.



Proposed by Jason Yang.

Answer: 2 (2π)

Solution: Each quarter-circle has radius 1. The perimeter of the arc of each quarter-circle is $\frac{1}{4} \times 2\pi \times 1$. We have 4 such arcs, so our final answer is 2π.

Set 2

Each problem in this set is worth 12 points.

L2.1) Rupert the Raven and Freddy the Falcon are fighting over the last banana. They will each roll a fair n-sided die ($n > 4$), whose faces are labeled with the integers from 1 to n. If the larger of the two rolls is at most 4, Freddy wins. Otherwise, Rupert wins. Find the value of n such that Rupert is 15 times as likely to win as Freddy.

Proposed by Jason Yang.

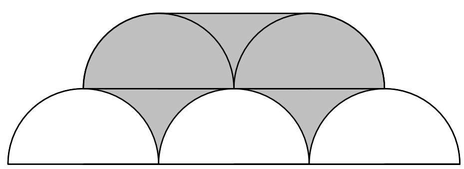
Answer: 16

Solution: The probability that Freddy wins is $\frac{4^2}{n^2}$, and the probability that Rupert wins is $1 - \frac{4^2}{n^2}$. Thus, we have

$$15 \times \frac{4^2}{n^2} = 1 - \frac{4^2}{n^2},$$
$$16 \times \frac{4^2}{n^2} = 1,$$
$$16 \times 16 = n^2,$$

so $n = 16$.

L2.2) The diagram below contains 5 congruent semicircles of radius 1. The shaded region can be expressed as $a - \frac{b}{c}\pi$, where a, b, and c are positive integers and b and c are relatively prime. Find $a + b + c$.



Proposed by Jason Yang.

Answer: $9 \left(6 - \frac{1}{2}\pi\right)$

Solution: The shaded region can be split into quarter circles and "spikes". Spikes are the region constructed from a square subtracted by a quarter circle. Rearranging shapes, the diagram consists of 4 quarter circles and 6 spikes. These form 4 squares of side length 1 and 2 spikes. The area of each spike is $1 - \frac{\pi}{4}$. The total area is then

$$4 + 2\left(1 - \frac{\pi}{4}\right) = 6 - \frac{1}{2}\pi$$

L2.3) Find the sum of the digits of

$$-20 \times -19 \times -18 \times \cdots \times 23 \times 24 \times 25.$$

Proposed by Ryan Bai.

Answer: 0

Solution: Note that the product includes 0, so the answer is just 0.

L2.4) Aidan, Bryan, Charles, and Dylan walk into a movie theater and randomly sit down in a row of 4 seats, with all seating arrangements being equally likely. Unfortunately, Bryan hates both Aidan and Charles. The probability Bryan does not sit next to Aidan or Charles can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Tiger Han.

Answer: 106 $\left(\frac{1}{6}\right)$

Solution: Since the order at which they take a seat doesn't matter, we can assume Bryan sits down first and Dylan sits down second. If Bryan sits in one of the two center seats, he is guaranteed to sit next to one of Aidan and Charles. Thus, Bryan must sit in one of the corner seats (occurs with probability $\frac{1}{2}$), and then Dylan must sit next to Bryan (occurs with probability $\frac{1}{3}$). Multiplying gives an answer of $\frac{1}{6}$.

Set 3

Each problem in this set is worth 14 points.

L3.1) Alice and Bob are workers that build walls at a uniform rate. Alice can build a certain wall in 3 hours. Alice and Bob together can build that same wall in 1 hour and 36 minutes. Find the number of hours it will take Bob to build a wall that is 14 times as wide and 15 times as tall as the original wall.

Proposed by Zhenghua Xie.

Answer: 720

Solution: Let r be the fraction of the original wall Bob can build in an hour. We have the following equation

$$\frac{1}{3} \times \frac{8}{5} + \frac{8}{5} r = 1.$$

Solving, we see that $r = \frac{7}{24}$, so Bob can build the oirginal wall in $\frac{24}{7}$ hours. Thus, the answer is $\frac{24}{7} \times 14 \times 15 = 720$ hours.

L3.2) Zhenghua writes the numbers $1, 2, 3, ..., 2025$ on a whiteboard. Zhenghua wants to select one or more of them such that, when Zhenghua multiplies his selected numbers together, their product is divisible by 5. The number of ways Zhenghua can do this can be expressed as $2^m - 2^n$, where m and n are positive integers. Find $100m + n$.

Proposed by Zhenghua Xie.

Answer: 204120 $\left(2^{2025} - 2^{1620}\right)$

Solution: We complementary count. The total number of ways for Zhenghua to select one or more numbers is $2^{2025} - 1$. The number of ways for Zhenghua to select one or more numbers and have their product not be divisible by 5 is $2^{1620} - 1$, since Zhenghua can't pick any number divisible by 5. Our answer is then $2^{2025} - 2^{1620}$.

L3.3) Let $ABCD$ be a rectangle. Let $CEFG$ be a square such that E lies on segment AB and F lies on segment CD. Given that $BE = DF$ and $DF + EF = 3$, the area of $ABCD$ can be expressed as $m - n\sqrt{2}$, where m and n are positive integers. Find $100m + n$.

Proposed by Aditya Bisain.

Answer: 8154 $\left(81 - 54\sqrt{2}\right)$

Solution: Note that CF is the diagonal of square $CEFG$, so $\triangle CEF$ is a 45-45-90 triangle. Let Y be the foot of the altitude from E to BC. We know $YC = YF = x$ and $FE = 3 - x$, so

$$3 - x = \sqrt{2}x,$$

which yields $x = \frac{3}{1+\sqrt{2}}$. Now, $BC = x$ and $AB = 3x$, so our final answer is

$$3x^2 = 81 - 54\sqrt{2}.$$

L3.4) Byan and Rai each pick a positive integer less than 100. It turns out that Byan's number is divisible by 5 and is greater than Rai's number. In addition, Rai's number divides Byan's number. Find the sum of all possible numbers Rai could've picked.

Proposed by Larry Wu.

Answer: 385

Solution: If Rai's number is divisible by 5, then Rai's number can be $5, 10, 15, \ldots, 45$ and Byan's number could be 2 times Rai's number. Otherwise, Rai's number must be less than or equal to 19 and Byan's number could be 5 times Rai's number.

Our answer is then

$$(1 + 2 + \cdots + 19) + (20 + 25 + \cdots + 45) = 190 + 190 = 380.$$

Set 4

Each problem in this set is worth 16 points.

L4.1) Let $P(x)$ be a polynomial of degree 2025, and let $Q(x)$ be a polynomial of degree 2. The graph of $Q(x)$ is tangent to the graph of $P(x)$ at at least 20×25 different points. Find the maximum possible number of total intersections (counting the tangencies) the graph of $P(x)$ could have with the graph of $Q(x)$.

Proposed by Rohan Mallick.

Answer: 1525

Solution: Consider the degree 2025 polynomial $R(x) = P(x) - Q(x)$. Note that each intersection of $P(x)$ and $Q(x)$ corresponds to a root of $R(x)$. $R(x)$ can have up to 2025 real roots, counting multiplicity. However, this over counts the $20 \times 25 = 500$ tangencies, which are double roots. Thus, $R(x)$ can have at most $2025 - 500 = 1525$ total roots, which equates to at most 1525 intersections between $P(x)$ and $Q(x)$.

An example of when they have 1525 intersections is when

$$P(x) = (x-1)^2 \cdot (x-2)^2 \cdot \dots \cdot (x-500)^2 \cdot (x-501) \cdot (x-502) \cdot \dots \cdot (x-1525) + x^2,$$

and $Q(x) = x^2$.

L4.2) Find the sum of all positive integers whose positive integer divisors (including itself) sum to 434.

Proposed by Aidan Bai.

Answer: 1210 $(208 + 244 + 325 + 433)$

Solution: The sum of the factors of a positive integer of the form

$$n = p_1^{e_1} \times p_2^{e_2} \times \dots \times p_m^{e_m}$$

is

$$(p_1^0 + p_1^1 + \dots + p_1^{e_1}) \times (p_2^0 + p_2^1 + \dots + p_2^{e_2}) \times \dots \times (p_m^0 + p_m^1 + \dots + p_m^{e_m}).$$

Thus, we want to express 434 in that form. 434 factorizes as $2 \times 7 \times 31$, so its factors are $1, 2, 7, 14, 31, 62, 217, 434$. For each factor x, we wish to find all possible ways of expressing it as

$$p^0 + p^1 + \dots + p^e.$$

We can now proceed by careful enumeration. There are not a lot of cases to check since for most values of p and e the above summation is already larger than 434. After enumerating, we obtain the following:

$$\begin{aligned}7 &= 2^0 + 2^1 + 2^2,\\ 14 &= 13^0 + 13^1,\\ 31 &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4,\\ 31 &= 5^0 + 5^1 + 5^2,\\ 62 &= 61^0 + 61^1,\\ 434 &= 433^0 + 433^1.\end{aligned}$$

Now, we can write 434 as

$$\begin{aligned}7 \times 62 &= \left(2^0 + 2^1 + 2^2\right) \cdot \left(61^0 + 61^1\right) && \Rightarrow n = 244,\\ 14 \times 31 &= \left(13^0 + 13^1\right) \cdot \left(2^0 + 2^1 + 2^2 + 2^3 + 2^4\right) && \Rightarrow n = 208,\\ 14 \times 31 &= \left(13^0 + 13^1\right) \cdot \left(5^0 + 5^1 + 5^2\right) && \Rightarrow n = 325,\\ 434 &= \left(433^0 + 433^1\right) && \Rightarrow n = 433,\end{aligned}$$

which yields all possible values of n.

L4.3) Rupert the Raven is playing a game. His score starts at 0. Every second, he flips a coin. If the coin lands heads, his score increases by 1. Otherwise, his score will reset to 0. Rupert repeats this for 100 coin flips. The expected number of times his score becomes 5 after a coin flip can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ryan Bai.

Answer: 9764 $\left(\frac{97}{64}\right)$

Solution: We use linearity of expectation. Let $f(x)$ be the expected number of times Rupert's score is at least x after a coin flip. Note that the expected number of times Rupert's score is 5 is $f(5) - f(6)$.

Now, for Rupert's score to be at least 5 after a coin flip, we just need the last 5 coin flips to all be heads. However, we need to ignore the first 4 coin flips. Thus, $f(5) = \frac{96}{32}$.

By a similar reasoning, $f(6) = \frac{95}{64}$, so our answer is

$$\frac{96}{32} - \frac{95}{64} = \frac{97}{64}.$$

Note: an alternative solution is to compute the expected number of times Rupert's score resets from some number at least 5. This is equivalent to computing the expected number of times Rupert flips, in this order, `HHHHHT`. We then need to account for Rupert ending the game with a score of at least 5.

L4.4) A right circular cone containing an infinite series of spheres $S_0, S_1, S_2, \ldots$ has a base of radius 3 and a height of 4. Let S_0 be the largest sphere that can be placed entirely within the cone, and for $n > 0$, let S_n be the largest sphere that can be placed entirely within the space between the tip of the cone and S_{n-1}. The total volume of all the spheres can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Zhenghua Xie.

Answer: 3207 $\left(\frac{32}{7}\pi\right)$

Solution: Instead of calculating the radii in 3D, we can consider a cross section of the cone by slicing directly from the tip downwards. This will result in a triangle with circles since the spheres are tangent to the surfaces of the cone. Shifting perspective from 3D to 2D, this sum becomes an infinite sequence of incircles.

Our isosceles triangle has a height of length 4, base of length 6, and legs of length 5. The inradius of this triangle is given by $r = \frac{A}{s} = \frac{12}{8} = \frac{3}{2}$. Now, drawing a line tangent to the top of the incircle and parallel to the base of our isosceles triangle reveals a similar triangle. The smaller triangle has a height of $4 - 3 = 1$, so its inradius is $\frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$. Thus, the radii of the spheres form the following infinite sequence: $\frac{3}{2}, \frac{3}{8}, \frac{3}{32}, \frac{3}{128}, \ldots$

To finish, we compute the sum of the infinite geometric series

$$\frac{4\pi}{3}\left[\left(\frac{3}{2}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{3}{32}\right)^3 + \cdots\right],$$

which comes out to $\frac{32\pi}{7}$.

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

Note: $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$ and $\lfloor 14 \rfloor = 14$.

L5.1) The mex, or minimum excluded value, of a list of nonnegative integers is defined as the smallest nonnegative integer that does not appear in the list. For example, the mex of $0, 9, 2, 1, 1, 0$ is 3, because 0, 1, and 2 appear, but 3 does not.

Submit a nonnegative integer X. If M is the mex of all submissions to this problem, you will receive $\left\lfloor \frac{20}{(M-X)^2} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

Proposed by Larry Wu.

Answer: The mex of all submissions was 7.

Solution: The Nash equilibria of this game are:
- everyone submits 1 so the mex is 0
- everyone submits either 0 or 2, with at least one person submitting 0 so the mex is 1

In theory, it is always at least as optimal to submit 1 instead of 0. In practice, it turns out we had 4 teams across both divisions submit 0. All valid submissions we received are below. The modal submission of 6 received full points.

0	0	0	0	1	1	1	1	1
2	2	2	3	3	4	4	4	5
5	5	6	6	6	6	6	6	6
9	9	10	12	13	16	17	18	19
22	22	23	26	26	29	32	34	49

L5.2) The sequence of Lucas numbers is defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Estimate the number of integers between 1 and 2025, inclusive, that can be expressed as the sum of 3 (not necessarily distinct) Lucas numbers.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot 0.98^{|A-E|} \right\rfloor$ points.

Proposed by Zhenghua Xie.

Answer: 469

Solution: First, let's list out the Lucas numbers under 2025.

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207.$$

Observe that expressing a number as the sum of one or two Lucas numbers generally implies it can be written as the sum of three Lucas numbers (some of which are necessarily consecutive), unless the number is < 3. Note that $L_0 + L_2 = L_1 + L_3$. Also note that the "not necessarily distinct" condition is useless because $2L_n = L_{n+1} + L_{n-2}$, and if $n < 2$, then we can trivially see $2L_1 = L_0$ and $2L_0 = L_3$. Then, the number of numbers less than 2025 expressable as the sum of one or two Lucas numbers is about $14 + \binom{15}{2} - 1 = 119$.

Consider the case for three Lucas numbers. If two of them are consecutive, then this is reducable to the two Lucas numbers case. Then, we just have to consider the number of ways to take 3 non-consecutive Lucas numbers from the first 16 Lucas numbers, which there are $\binom{14}{3} = 364$ ways to do. Note that exactly one of these, $L_{15} + L_{13} + L_{11}$ is greater than 2025, so we will subtract one from the total. Again, note that $L_0 + L_2 = L_1 + L_3$, so another 12 should be subtracted from the total.

Summing it all up, we get approximately $119 + 364 - 1 - 12 = 469$ as our estimate, which happens to be exactly correct.

L5.3) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Consider writing down $F_0, F_1, ..., F_{10000}$ and joining them all together. Estimate the number of times the digits $1, 2, 3$ occur consecutively in any order. For example, joining $F_0, F_1, ..., F_{10}$ gives 11235813213455, which contains 4 such occurrences.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot \min\left(\frac{A}{E}, \frac{E}{A}\right)^{13} \right\rfloor$ points.

Proposed by Sepehr Golsefidy.

Answer: 62571

Solution:

Note: This solution makes use of logarithms, which wasn't supposed to show up in Division II– sorry about that. It is technically possible to get rid of all the logarithms and rephrasing it in terms of powers, but that makes everything a lot more awkward.

We can assume the digits of the resulting string is random. Each consecutive block of 3 characters has a $\frac{6}{1000}$ chance of containing the digits 1, 2, and 3. Thus, if we estimate M to be the length of the final string, we can approximate our answer as $\frac{6}{1000}M$.

Let L_i be the number of digits in the decimal expansion of F_i, and let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. By the properties of the Fibonacci recurrence, we have $F_i \approx \varphi^i$, and

$$L_i = 1 + \lfloor \log_{10}(F_i) \rfloor \approx \log_{10}(\varphi^i) = i \log_{10}(\varphi).$$

Note that the above approximations are all very rough and are off by one in places, but it doesn't matter in the long run.

Thus,

$$M = L_0 + L_1 + \cdots + L_{10000} \approx (0 + 1 + \cdots + 10000) \log_{10}(\varphi) \approx 5 \times 10^7 \log_{10}(\varphi).$$

Now, we approximate $\log_{10}(\varphi) \approx \log_{10}(1.6) = 4 \log_{10}(2) - 1$. Now, $2^{10} \approx 10^3$, so $\log_{10}(2) \approx \frac{3}{10}$. Plugging this back in gives $\log_{10}(\varphi) \approx 0.2$ (note: the real value is 0.208987). Our final answer is

$$\frac{6}{1000} \times 5 \times 10^7 \times 0.2 = 60000.$$

More accurate approximations of $\log_{10}(\varphi)$ will achieve higher scores.

L5.4) Submit an integer between 1 and 240, inclusive. Your *index* is initially 0. Every positive integer divisor x of your submission will add $\frac{1}{n}$ points to your index, where n is the number of submissions to this problem (including yours) that are divisible by x.

If your index is S and the maximum index amongst all teams is M, you will receive $\left\lfloor 20 \cdot \left(\frac{S}{M}\right)^{1.5} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

Proposed by Rohan Mallick.

Answer: Here are all the valid submissions we received:
2 6 7 24 24 26 40 69 77 120 120 128 144 147 154 180 192 200 209 209 210 210 210 210 210 212 212 215 221 224 230 234 240 240 240 240 240 240 240 240 240 240

Solution: Here's a list of the best numbers to submit against the pool of submissions we received during the contest. Note that only the top two submissions would have beaten the previous best answer of $234 = 2 \times 3^2 \times 13$, which received an index of 5.63. Also of note is that the next three best submissions, $224 = 2^5 \times 7$ with an index of 4.79, $180 = 2^2 \times 3^2 \times 5$ with an index of 4.75, and $200 = 2^3 \times 5^2$ with an index of 4.38 are still better than some of the entries in this list.

Number	**Index**
$204 = 2^2 \times 3 \times 17$	5.73
$216 = 2^3 \times 3^3$	5.67
$228 = 2^2 \times 3 \times 19$	5.56
$198 = 2 \times 3^2 \times 11$	5.33
$220 = 2^2 \times 5 \times 11$	4.94
$132 = 2^2 \times 3 \times 11$	4.93
$162 = 2 \times 3^4$	4.63
$232 = 2^3 \times 29$	4.14
$222 = 2 \times 3 \times 37$	4.13
$186 = 2 \times 3 \times 31$	4.13

CCA Math Bonanza

March 8, 2025

Division II Tiebreaker Round Solutions

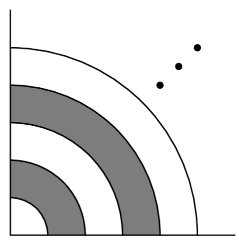
TB1) Find the number of factors of 2025 that are multiples of 5 but not 27.

Proposed by Aditya Bisain.

Answer: 6

Solution: Prime factors of 2025 are of the form $3^a \times 5^b$, where $0 \leq a \leq 4$ and $0 \leq b \leq 2$. The divisibility condition additionally gives $b \geq 1$ and $a < 3$. We have 2 choices for b and 3 choices for a, for an answer of $2 \times 3 = 6$.

TB2) Shown below are 100 quarter circles of radii $1, 2, 3, \ldots, 100$, all of which share the same center and are oriented in the same direction. These circles divide the figure into 100 rings. The first ring is colored white, and subsequent rings alternate between grey and white. The area of the grey region can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.



Proposed by Zhenghua Xie.

Answer: 252502 $\left(\frac{2525}{2}\pi\right)$

Solution: The area of the first shaded ring is $\frac{1}{4}(2^2 - 1^2)\pi = \frac{1}{4}(3\pi)$. The area of the next shaded ring is $\frac{1}{4}(4^2 - 3^2)\pi = \frac{1}{4}(7\pi)$. Our total area is

$$\begin{aligned} &\frac{1}{4}(2^2 - 1^2 + 4^2 - 3^2 + \cdots + 100^2 - 99^2)\pi \\ &= \frac{1}{4}(3 + 7 + 11 + \cdots + 199)\pi \\ &= \frac{2525}{2}\pi. \end{aligned}$$

TB3) Let n and x be positive integers such that the sum of the digits of x is 2 and

$$9900\,9900\,9900\,9901 \times n = x.$$

Find the smallest such n.

Proposed by Alex Backues.

Answer: 101

Solution: Note that since the digits of x sum to 2, x is either of the form $2 \cdot 10^k$ or 100...000100... However, since x is divisible by 9900 9900 9900 9901, it can't be of the form $2 \cdot 10^k$.

x starts with a 1, so n must start with a 1 or 2, since all other digits will cause the leading digit of x to not be 1. Furthermore, x's units digit is either 0 or a 1, so n's units digit must also be either 0 or 1. Checking all numbers of this form reveals that 101 is the smallest solution.

TB4) Charlotte the cat lives in Cartesia, a city on the coordinate plane whose roads are the lines $x = a$ and $y = a$ for integers a. Charlotte is currently standing at the origin, and would like to walk to the highway at $y = 5$, and then to her home at $(6, 1)$. Let D be the length of the shortest possible path she could take home. Find the distinct number of paths of length D Charlotte can take.

Proposed by Rohan Mallick.

Answer: 5005

Solution: Note that for any point P on $y = 5$, the distance from P to $(6, 1)$ is the same as the distance from P to $(6, 9)$, because $(6, 1)$ and $(6, 9)$ are reflections across $y = 5$. Thus, we can instead consider journeys from $(0, 0)$ to $y = 5$ to $(6, 9)$. However, journeys with the shortest possible path from $0, 0$ to $y = 5$ to $(6, 9)$ are equivalent to journeys with the shortest possible path from $(0, 0)$ to $(6, 9)$, which means the answer is $\binom{15}{6} = 5005$.