CCA Math Bonanza

March 8, 2025

Division II Lightning Round

Set 1

Each problem in this set is worth 10 points.

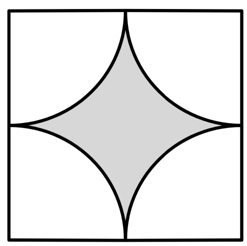
L1.1) Find the number of points (x, y) on the coordinate plane such that x and y are positive integers that are at most 9, and the units digit of xy is 7.

L1.2) Three numbers have a max of 27, a min of 13, and a mean of 21. Find their median.

L1.3) Find the remainder when the following expression is divided by 2025.

$$2^{0^{2^{5^{2^{0^{2^{5^{\cdots}}}}}}}}$$

L1.4) The diagram below contains 4 congruent quarter-circles. The side length of the outer square is 2. The perimeter of the shaded region can be expressed as $n\pi$, where n is a positive integer. Find n.

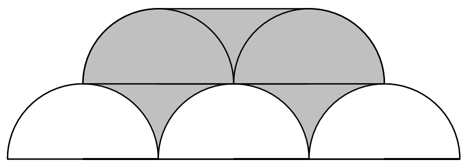


Set 2

Each problem in this set is worth 12 points.

L2.1) Rupert the Raven and Freddy the Falcon are fighting over the last banana. They will each roll a fair n-sided die ($n > 4$), whose faces are labeled with the integers from 1 to n. If the larger of the two rolls is at most 4, Freddy wins. Otherwise, Rupert wins. Find the value of n such that Rupert is 15 times as likely to win as Freddy.

L2.2) The diagram below contains 5 congruent semicircles of radius 1. The shaded region can be expressed as $a - \frac{b}{c}\pi$, where a, b, and c are positive integers and b and c are relatively prime. Find $a + b + c$.



L2.3) Find the sum of the digits of

$$-20 \times -19 \times -18 \times \cdots \times 23 \times 24 \times 25.$$

L2.4) Aidan, Bryan, Charles, and Dylan walk into a movie theater and randomly sit down in a row of 4 seats, with all seating arrangements being equally likely. Unfortunately, Bryan hates both Aidan and Charles. The probability Bryan does not sit next to Aidan or Charles can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Set 3

Each problem in this set is worth 14 points.

L3.1) Alice and Bob are workers that build walls at a uniform rate. Alice can build a certain wall in 3 hours. Alice and Bob together can build that same wall in 1 hour and 36 minutes. Find the number of hours it will take Bob to build a wall that is 14 times as wide and 15 times as tall as the original wall.

L3.2) Zhenghua writes the numbers $1, 2, 3, \ldots, 2025$ on a whiteboard. Zhenghua wants to select one or more of them such that, when Zhenghua multiplies his selected numbers together, their product is divisible by 5. The number of ways Zhenghua can do this can be expressed as $2^m - 2^n$, where m and n are positive integers. Find $100m + n$.

L3.3) Let $ABCD$ be a rectangle. Let $CEFG$ be a square such that E lies on segment AB and F lies on segment CD. Given that $BE = DF$ and $DF + EF = 3$, the area of $ABCD$ can be expressed as $m - n\sqrt{2}$, where m and n are positive integers. Find $100m + n$.

L3.4) Byan and Rai each pick a positive integer less than 100. It turns out that Byan's number is divisible by 5 and is greater than Rai's number. In addition, Rai's number divides Byan's number. Find the sum of all possible numbers Rai could've picked.

Set 4

Each problem in this set is worth 16 points.

L4.1) Let $P(x)$ be a polynomial of degree 2025, and let $Q(x)$ be a polynomial of degree 2. The graph of $Q(x)$ is tangent to the graph of $P(x)$ at at least 20×25 different points. Find the maximum possible number of total intersections (counting the tangencies) the graph of $P(x)$ could have with the graph of $Q(x)$.

L4.2) Find the sum of all positive integers whose positive integer divisors (including itself) sum to 434.

L4.3) Rupert the Raven is playing a game. His score starts at 0. Every second, he flips a coin. If the coin lands heads, his score increases by 1. Otherwise, his score will reset to 0. Rupert repeats this for 100 coin flips. The expected number of times his score becomes 5 after a coin flip can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L4.4) A right circular cone containing an infinite series of spheres $S_0, S_1, S_2, \ldots$ has a base of radius 3 and a height of 4. Let S_0 be the largest sphere that can be placed entirely within the cone, and for $n > 0$, let S_n be the largest sphere that can be placed entirely within the space between the tip of the cone and S_{n-1}. The total volume of all the spheres can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

Note: $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$ and $\lfloor 14 \rfloor = 14$.

L5.1) The mex, or minimum excluded value, of a list of nonnegative integers is defined as the smallest nonnegative integer that does not appear in the list. For example, the mex of $0, 9, 2, 1, 1, 0$ is 3, because 0, 1, and 2 appear, but 3 does not.

Submit a nonnegative integer X. If M is the mex of all submissions to this problem, you will receive $\left\lfloor \frac{20}{(M-X)^2} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

L5.2) The sequence of Lucas numbers is defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Estimate the number of integers between 1 and 2025, inclusive, that can be expressed as the sum of 3 (not necessarily distinct) Lucas numbers.

Submit a positive integer E. If the correct answer is A, you will receive $\lfloor 20 \cdot 0.98^{|A-E|} \rfloor$ points.

L5.3) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Consider writing down $F_0, F_1, F_2, \ldots, F_{10000}$ and joining them all together. Estimate the number of times the digits $1, 2, 3$ occur consecutively in any order. For example, joining $F_0, F_1, F_2, \ldots, F_{10}$ gives 11235813213455, which contains 4 such occurrences.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot \min\left(\frac{A}{E}, \frac{E}{A}\right)^{13} \right\rfloor$ points.

L5.4) Submit an integer between 1 and 240, inclusive. Your *index* is initially 0. Every positive integer divisor x of your submission will add $\frac{1}{n}$ points to your index, where n is the number of submissions to this problem (including yours) that are divisible by x.

If your index is S and the maximum index amongst all teams is M, you will receive $\left\lfloor 20 \cdot \left(\frac{S}{M}\right)^{1.5} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.