CCA Math Bonanza

March 8, 2025

Division II Individual Round

I1) Rupert the Raven just won the lottery in CCA land! However, the government will take 20% of his winnings as tax. After paying taxes, Rupert wants to have exactly 1 thousand dollars. Find the amount of money Rupert needs to have won.

I2) When Lukas turned 7 years old, his mother told him that she was 5 times as old as him. Find the age of his mother when Lukas turns 60 years old.

I3) Find the value of

$$\sqrt{(2-0-2-5)(2+0+2+5)(2-0+2-5)(2+0-2+5)}.$$

I4) Find the number of points (x, y) on the coordinate plane such that x and y are positive integers and $x + y \leq 7$.

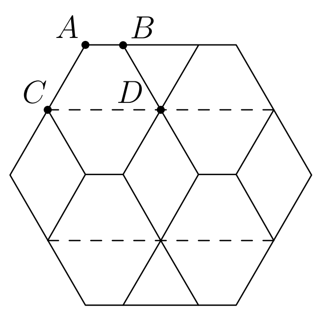
I5) Let $a \heartsuit b$ be the result of dividing a by b and keeping only the integer part. For example, $11 \heartsuit 3 = 3$ because $\frac{11}{3} = 3\,\frac{2}{3}$, and $29 \heartsuit 7 = 4$ because $\frac{29}{7} = 4\,\frac{1}{7}$. Find $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$.

I6) Byan picks a random integer between 0 and 4, inclusive. Then, Rai flips 4 fair coins. The probability the amount of heads Rai lands is equal to Byan's integer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

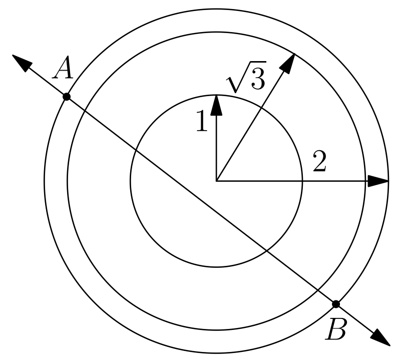
I7) The sum of the coefficients (including the constant term) of the polynomial $\left(x^2 + ax - 4\right)^2$ is 36. Find the sum of the squares of all possible values of a.

I8) Let y be an integer, and let $x = 5^4 - 5^2y^2$. Given that x is an even positive integer, find the largest possible number of positive divisors of x.

I9) The diagram below shows eight congruent isosceles trapezoids arranged inside a regular hexagon. $\frac{AB}{CD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.



I10) In the diagram below, there are three circles of radii 1, $\sqrt{3}$, and 2, all with the same center. Points A and B are chosen uniformly at random along the circumference of the outermost circle. In the diagram, the line AB intersects the circle six times. The probability that line AB intersects the circles exactly four times can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.



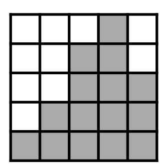
I11) Let $f(n)$ denote the number of 1s in the base 2 representation of n. For example, $f(5) = 2$. Let $s(n) = f(1) + f(2) + \cdots + f(n)$. Find $s(2^0) + s(2^1) + \cdots + s(2^9)$.

I12) Let P and Q be fixed points such that $PQ = 12$. Let A and B be points such that P lies on AB and $AQ : QB : BA = 3 : 4 : 5$. Find the maximum area of $\triangle AQB$ over all valid locations of A and B.

I13) An n-tower is constructed by shading cells in an $n \times n$ grid according to the following rules:

- The kth row from the top has k consecutive shaded cells.
- Each shaded cell not in the bottom row is directly above another shaded cell.

An example of a 5-tower is shown below.



Let A be the number of 25-towers such that the middle cell of the 1st row from the top and the leftmost cell of the 21st row from the top are shaded. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

I14) Let ℓ be a non-vertical line that does not intersect the curve $y = \frac{x^4 - x^3}{x - 1}$. Let A be the smallest real number which is larger than all possible slopes of such a line ℓ. A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I15) Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 5$, and $CA = 4$. Let D and E be points on segments BC and AC, respectively. The minimum value of $AD + DE + EB$ over all choices of D and E can be expressed as $\frac{a\sqrt{b}}{c}$, where b is a positive integer not divisible by the square of any prime, and a and c are relatively prime positive integers. Find $a + b + c$.