CCA Math Bonanza

March 8, 2025

Division I Solutions

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Division I Individual Round

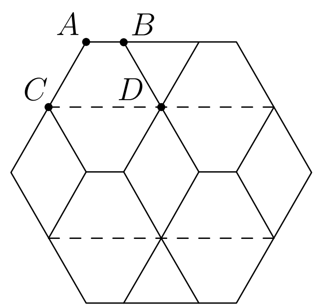
I1) Let $a \heartsuit b$ be the result of dividing a by b and keeping only the integer part. For example, $11 \heartsuit 3 = 3$ because $\frac{11}{3} = 3\,\frac{2}{3}$, and $29 \heartsuit 7 = 4$ because $\frac{29}{7} = 4\,\frac{1}{7}$. Find $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$.

I2) Find $147 \times \left(67^2 - 64^2\right) + 3^2 \times 23 \times 147$.

I3) The sum of the coefficients (including the constant term) of the polynomial $\left(x^2 + ax - 4\right)^2$ is 36. Find the sum of the squares of all possible values of a.

I4) Let y be an integer, and let $x = 5^4 - 5^2 y^2$. Given that x is an even positive integer, find the largest possible number of positive divisors of x.

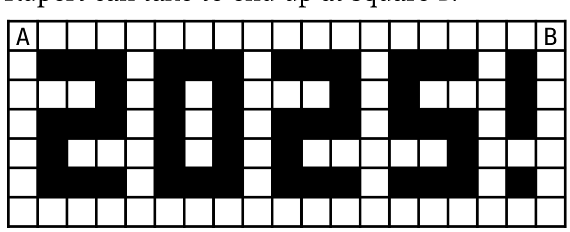
I5) The diagram below shows eight congruent isosceles trapezoids arranged inside a regular hexagon. $\frac{AB}{CD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.



I6) Byan picks a random integer between 0 and 3, inclusive. Then, Rai flips 4 fair coins. The probability that the amount of heads Rai lands is equal to Byan's integer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I7) Let $\triangle ABC$ be a triangle with $AB = 14$, $BC = 9$, and $\angle BAC = 15°$. Find the sum of all possible values of the area of $\triangle ABC$.

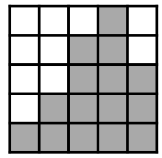
I8) Rupert starts on square A and wants to reach square B. He can repeatedly move to any white square that shares a side with his current square, but he cannot revisit any square he has already moved to. Find the number of paths Rupert can take to end up at square B.



I9) An n-tower is constructed by shading cells in an $n \times n$ grid according to the following rules:

- The kth row from the top has k consecutive shaded cells.
- Each shaded cell not in the bottom row is directly above another shaded cell.

An example of a 5-tower is shown below.



Let A be the number of 25-towers such that the middle cell of the 1st row from the top and the leftmost cell of the 21st row from the top are shaded. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

I10) Let ℓ be a non-vertical line that does not intersect the curve $y = \frac{x^4 - x^3}{x - 1}$. Let A be the smallest real number which is larger than all possible slopes of such a line ℓ. A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I11) Let $\triangle ABC$ be a triangle with $BC = 32$ and circumradius 18. Let O be the circumcenter of $\triangle ABC$, and let D be the foot of the altitude from A to BC. Suppose that the area of $\triangle ABC$ is 3 times the area of $\triangle OBC$. Find OD^2.

I12) Let $a_1, a_2, ..., a_8$ be a permutation of $2, 3, 3, 5, 6, 6, 7, 8$. a_i is considered a *prefix maximum* if either $i = 1$ or $a_i > \max(a_1, a_2, ..., a_{i-1})$. Define the *conspiratorial value* of a permutation to be the product of its prefix maximums. Let A be the sum of the conspiratorial values over all $\frac{8!}{2!2!}$ permutations. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

I13) Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 7$, $CA = 5$, and $\angle BAC = 120°$. Let P be a point inside $\triangle ABC$ such that $AP = 1$, and $\angle BPC = 150°$. Let the tangent line to the cirumcircle of $\triangle BPC$ at P intersect line BC at X, and let line AP intersect line BC at Y. The length XY can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I14) Let x be the unique positive real number satisfying

$$x^{x^{x^{27}+27}} = 81^{3^{13}}.$$

Find the smallest integer n such that $x^n \geq 3^{64}$.

I15) Let

$$X = \sum_{i=1}^{2025} i^{2025}.$$

Let m be the largest integer such that 3^m divides X, and let n be the largest integer such that 5^n divides X. Find $m \times n$.

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Division I Team Round

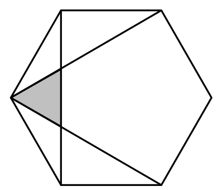
T1) Larry is making a burger. He starts with a bun and can add any combination of lettuce, cheese, tomato, onion, and meat, including the option to leave the burger plain with just the bun. However, he doesn't want to make a burger that has meat but no cheese. The order in which he adds his ingredients doesn't matter. Find the number of different burgers Larry can create.

T2) The value

$$\frac{25!^2 - 24!^2}{25!^2 + 24!^2}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

T3) The regular hexagon shown in the figure below has side length 1. The area of the shaded region can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.



T4) The maximum possible value of $\frac{t}{3^t} - \left(\frac{t}{3^{t-1}}\right)^2$ over all real numbers t can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

T5) Find the number of positive integers n such that $n \leq 750$ and $\gcd(1500, n)$ is a prime number.

T6) Three points A, B, and C are chosen uniformly at random along the circumference of a circle of radius 1. The probability that all angles in $\triangle ABC$ are less than $75°$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

T7) Let a, b, and c be real numbers that satisfy the following system of equations:

$$(a+1)(b+1)(c+1) - abc = 4$$
$$(a-1)(b-1)(c-1) + ab + bc + ca = 4$$
$$a + b + c + \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = 0$$

The sum of all possible values of $a^2 + b^2 + c^2$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

T8) Let $\triangle ABC$ be a triangle with $BC = 12$, $\angle BAC = 60°$, and $\angle ABC = 75°$. Let O, H, and I be the circumcenter, orthocenter, and incenter of $\triangle ABC$, respectively. If O' is the reflection of O over BC, then the area of $O'HIO$ can be expressed as $\sqrt{v} - \sqrt{w}$, where v and w are positive integers. Find $100v + w$.

T9) Call a function $f(x)$ which takes in rational numbers as inputs and outputs real numbers *awesome* if, for any rational number x, $f(x) = f(2x) + f\left(\frac{x}{2}\right)$, and for any odd integer k, $f(k) = k$. Let S be a subset of $\{1, 2, ..., 100\}$ such that $\sum_{n \in S} f(n)$ has exactly one possible value across all *awesome* functions f. Find the maximum possible size of S.

T10) Let $\triangle ABC$ be a triangle with $BC = 120$ and $\angle BAC = 60°$. Let M be the midpoint of BC. It is given that $AM = 90$. Let Ω be the circumcircle of $\triangle ABC$, and let D be the midpoint of major arc BC. Let line AM intersect Ω at E, line DE intersect BC at F, and line AF intersect Ω at G. Let $K \neq G$ be the intersection of Ω with the circumcircle of MFG. Find AK.

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Division I Lightning Round

Set 1

Each problem in this set is worth 10 points.

L1.1) Aidan, Bryan, Charles, and Dylan walk into a movie theater and randomly sit down in a row of 4 seats, with all seating arrangements being equally likely. Unfortunately, Bryan hates both Aidan and Charles. The probability Bryan does not sit next to Aidan or Charles can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L1.2) Rupert the Raven and Freddy the Falcon are fighting over the last banana. They will each roll a fair n-sided die ($n > 4$), whose faces are labeled with the integers from 1 to n. If the larger of the two rolls is at most 4, Freddy wins. Otherwise, Rupert wins. Find the value of n such that Rupert is 15 times as likely to win as Freddy.

L1.3) Find the value of

$$(2025 \times 2022 - 2024 \times 2023) + (2021 \times 2018 - 2020 \times 2019) + \cdots + (5 \times 2 - 4 \times 3) + 1.$$

L1.4) Let $ABCD$ be a rectangle. Let $CEFG$ be a square such that E lies on segment AB and F lies on segment CD. Given that $BE = DF$ and $DF + EF = 3$, the area of $ABCD$ can be expressed as $m - n\sqrt{2}$, where m and n are positive integers. Find $100m + n$.

Set 2

Each problem in this set is worth 12 points.

L2.1) Byan and Rai each pick a positive integer less than 100. It turns out that Byan's number is divisible by 5 and is greater than Rai's number. In addition, Rai's number divides Byan's number. Find the sum of all possible numbers Rai could've picked.

L2.2) Rupert the Raven has 4 independent, weighted coins, whose probabilities of landing heads on a flip are 10%, 40%, 55%, and p, where $0 \leq p \leq 1$. If he flips all 4 coins simultanenously, the probability he lands an odd number of heads is exactly 50%. The sum of all possible values of p can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L2.3) A right circular cone containing an infinite series of spheres $S_0, S_1, S_2, \ldots$ has a base of radius 3 and a height of 4. Let S_0 be the largest sphere that can be placed entirely within the cone, and for $n > 0$, let S_n be the largest sphere that can be placed entirely within the space between the tip of the cone and S_{n-1}. The total volume of all the spheres can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.

L2.4) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers such that

$$a_1 = a_2 = \cdots = a_{100} = 1,$$

and for all $n > 100$,

$$a_n = a_{n-100} + a_{n-99} + \cdots + a_{n-1}.$$

Find the second smallest $n > 1$ such that $a_n \neq 2a_{n-1} - 1$.

Set 3

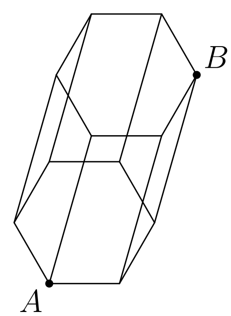
Each problem in this set is worth 14 points.

L3.1) There exists a unique 5-digit palindrome of the form $\overline{abcba}$ that, when multiplied by $\frac{3}{4}$, becomes $\overline{a0cbc}$. Find $\overline{abcba}$.

L3.2) The repeating decimal $0.\overline{ab}_k$ in base k can be written as $\frac{227}{306}$. For the minimum possible value of a, find $100a + b$.

L3.3) Rupert the Raven is playing a game. His score starts at 0. Every second, he flips a coin. If the coin lands heads, his score increases by 1. Otherwise, his score will reset to 0. Rupert repeats this for 100 coin flips. The expected number of times his score becomes 5 after a coin flip can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L3.4) Shown below is a right regular hexagonal prism whose hexagonal base has side length 1. Larry the ant is currently standing at point A and wishes to reach point B. He notes that there are more than two distinct paths along the surface of the prism he can take to minimize the total distance he travels. The maximum possible height of the prism can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.



Set 4

Each problem in this set is worth 16 points.

L4.1) Find the sum of $x + y$ over all pairs of positive integers x and y satisfying

$$x^2 + y^2 = 193(x - y).$$

L4.2) Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$, and let its incircle touch BC, CA, and AB at D, E, and F, respectively. Let I be the incenter of $\triangle ABC$. Let X be the intersection of lines EF and BC, and let Y be the intersection of lines XI and AD. $\frac{YE}{YF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L4.3) Find the value of

$$2048 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \frac{1}{xy(x+2)(x+y+4)}.$$

L4.4) Let

$$A = \sum_{n=0}^{12} n^2 \binom{12}{n}^2.$$

Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

L5.1) The mex, or minimum excluded value, of a list of nonnegative integers is defined as the smallest nonnegative integer that does not appear in the list. For example, the mex of $0, 9, 2, 1, 1, 0$ is 3, because 0, 1, and 2 appear, but 3 does not.

Submit a nonnegative integer X. If M is the mex of all submissions to this problem, you will receive $\left\lfloor \frac{20}{(M-X)^2} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

L5.2) Let N be the largest positive integer such that no substring of its decimal expansion is divisible by 193. Estimate the sum of the squares of the digits of N.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot \min\left(\frac{A}{E}, \frac{E}{A}\right)^{13} \right\rfloor$ points.

Hint: 193 is a prime. The repeating part of the decimal expansion of $\frac{1}{193}$ has period 192.

L5.3) The sequence of Lucas numbers is defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Estimate the number of integers between 1 and 2025, inclusive, that can be expressed as the sum of 3 (not necessarily distinct) Lucas numbers.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot 0.98^{|A-E|} \right\rfloor$ points.

L5.4) Submit an integer between 1 and 240, inclusive. Your *index* is initially 0. Every positive integer divisor x of your submission will add $\frac{1}{n}$ points to your index, where n is the number of submissions to this problem (including yours) that are divisible by x.

If your index is S and the maximum index amongst all teams is M, you will receive $\left\lfloor 20 \cdot \left(\frac{S}{M}\right)^{1.5} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

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Division I Tiebreaker Round

TB1) Let n and x be positive integers such that the sum of the digits of x is 2 and

$$9900\,9900\,9900\,9901 \times n = x.$$

Find the smallest such n.

TB2) Charlotte the cat lives in Cartesia, a city on the coordinate plane whose roads are the lines $x = a$ and $y = a$ for integers a. Charlotte is currently standing at the origin, and would like to walk to the highway at $y = 5$, and then to her home at $(6, 1)$. Let D be the length of the shortest possible path she could take home. Find the distinct number of paths of length D Charlotte can take.

TB3) A positive integer n is *x-central* if there exists integers a and $-2 \leq b \leq 4$ such that $n = ax + b$. Find the number of positive integers $k < 8390$ that satisfy exactly 2 of the following:

- k is 13-central
- k is 17-central
- k is 19-central

TB4) Unit cube $ABCD - EFGH$ has square faces $ABCD$ and $EFGH$, with vertices A, B, C, D adjacent to vertices E, F, G, H, respectively. Two regular tetrahedrons with bases DBE and DBG are constructed. Let the apexes of the two tetrahedrons be P and Q, with both P and Q lying outside the cube. The length PQ can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.

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Division I Individual Round Answer Key

I1) 2

I2) 88200

I3) 90

I4) 24

I5) 103

I6) 1564

I7) 49

I8) 56

I9) 59

I10) 304

I11) 120

I12) 48

I13) 4908

I14) 432

I15) 55

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Division I Team Round Answer Key

T1) 24

T2) 31513

T3) 148

T4) 136

T5) 240

T6) 116

T7) 9304

T8) 86688

T9) 67

T10) 100

CCA Math Bonanza

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Division I Lightning Round Answer Key

Set 1

Each problem in this set is worth 10 points.

L1.1) 106

L1.2) 16

L1.3) -1011

L1.4) 8154

Set 2

Each problem in this set is worth 12 points.

L2.1) 385

L2.2) 102

L2.3) 3207

L2.4) 202

Set 3

Each problem in this set is worth 14 points.

L3.1) 27972

L3.2) 2533

L3.3) 9764

L3.4) 103

Set 4

Each problem in this set is worth 16 points.

L4.1) 263

L4.2) 5245

L4.3) 576

L4.4) 79

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

L5.1) The mex of all submissions was 7.

L5.2) 13175

L5.3) 469

L5.4) Here are all the valid submissions we received:
2 6 7 24 24 26 40 69 77 120 120 128 144 147 154 180 192 200 209 209 210 210 210 210 210 212 212 215 221 224 230 234 240 240 240 240 240 240 240 240 240 240

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Division I Tiebreaker Round Answer Key

TB1) 101

TB2) 5005

TB3) 2744

TB4) 5009

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Division I Individual Round Solutions

I1) Let $a \heartsuit b$ be the result of dividing a by b and keeping only the integer part. For example, $11 \heartsuit 3 = 3$ because $\frac{11}{3} = 3\,\frac{2}{3}$, and $29 \heartsuit 7 = 4$ because $\frac{29}{7} = 4\,\frac{1}{7}$. Find $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$.

Proposed by Jason Yang.

Answer: 2

Solution: We wish to compute

$$\frac{\frac{2025}{3}}{\frac{2024}{7}},$$

but keeping only the integer part of each division. Since 2025 and 2024 are so close to each other, we can simplify $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$ to $7 \heartsuit 3 = 2$.

I2) Find $147 \times \left(67^2 - 64^2\right) + 3^2 \times 23 \times 147$.

Proposed by Zhenghua Xie.

Answer: 88200

Solution:

$$\begin{aligned} & 147 \times \left(67^2 - 64^2\right) + 3^2 \times 23 \times 147 \\ &= 147 \times (67 - 64) \times (67 + 64) \times 147 \times 3 \times 69 \\ &= 147 \times 3 \times (121 + 69) \\ &= 147 \times 3 \times 200 \\ &= 88200. \end{aligned}$$

I3) The sum of the coefficients (including the constant term) of the polynomial $\left(x^2 + ax - 4\right)^2$ is 36. Find the sum of the squares of all possible values of a.

Proposed by Aditya Bisain.

Answer: 90 ($a = 9$ or $a = -3$)

Solution: Let $P(x) = \left(x^2 + ax - 4\right)^2$. The sum of the coefficients of P is $P(1)$. Thus, we have

$$P(1) = (a - 3)^2 = 36,$$

so $a - 3 = \pm 6$, yielding solutions of $a = 9$ and $a = -3$.

I4) Let y be an integer, and let $x = 5^4 - 5^2y^2$. Given that x is an even positive integer, find the largest possible number of positive divisors of x.

Proposed by Aditya Bisain.

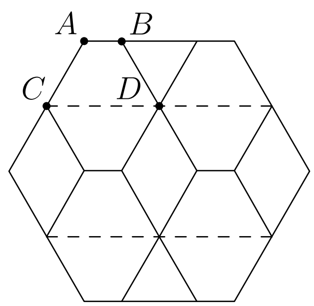
Answer: 24

Solution: We have $x = 5^2(5^2 - y^2)$. Since x is positive, we have $|y| < 5$. Furthermore, x is even, so y must be odd. Thus, $|y|$ is either 1 or 3.

- When $|y| = 1$, $x = 25 \times 24 = 2^3 \times 3 \times 5^2$ which has $4 \times 2 \times 3 = 24$ factors.
- When $|y| = 3$, $x = 25 \times 16 = 2^4 \times 5^2$ which has $5 \times 3 = 15$ factors.

Taking the larger of the two gives our final answer of 24.

I5) The diagram below shows eight congruent isosceles trapezoids arranged inside a regular hexagon. $\frac{AB}{CD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.



Proposed by Jason Yang.

Answer: 103 $\left(\frac{1}{3}\right)$

Solution: Without loss of generality, let the side length of the hexagon be 4. Then, $AC = BD = \frac{4}{2} = 2$. Let E be the foot of the altitude from A to CD and F be the foot of the altitude from B to CD. Since ACE is a 30-60-90 triangle, $CE = \frac{1}{2}AC = 1$, and $FD = \frac{1}{2}BD = 1$. Thus, $CD = CE + EF + FD = 1 + 1 + 1 = 3$. Also, the smaller triangle at the top of the hexagon is an equilateral triangle of side length 2, so $AB = 1$. Our answer is then $\frac{AB}{CD} = \frac{1}{3}$.

I6) Byan picks a random integer between 0 and 3, inclusive. Then, Rai flips 4 fair coins. The probability that the amount of heads Rai lands is equal to Byan's integer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ryan Bai.

Answer: 1564 $\left(\frac{15}{64}\right)$

Solution: If all 4 coins land heads, then the probability that Byan chooses the correct integer is 0 because Byan can only pick from 0 to 3. However, if the coins have anywhere from 0 to 3 heads, inclusive, the probability Byan picks the right number is just $\frac{1}{4}$. Therefore, since there is a $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ chance of landing all heads, there is a $1 - \frac{1}{16} = \frac{15}{16}$ chance of landing anything except all heads. Thus, the answer is $\left(\frac{15}{16}\right)\left(\frac{1}{4}\right) = \frac{15}{64}$.

I7) Let $\triangle ABC$ be a triangle with $AB = 14$, $BC = 9$, and $\angle BAC = 15°$. Find the sum of all possible values of the area of $\triangle ABC$.

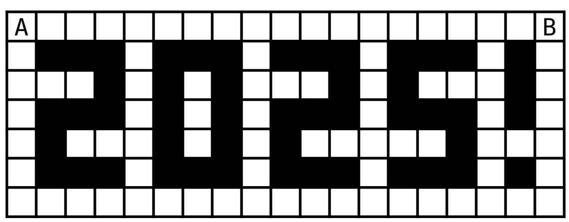
Proposed by Rohan Mallick.

Answer: 49

Solution: Let's start by constructing all the configurations of $\triangle ABC$. Let A and B be points such that $AB = 14$. Now, construct a ray starting at A which makes an angle of $15°$ with AB (there are two such rays but we end up with the same triangles up to a reflection abaout AB). Now, construct a circle centered at B with radius 9. C can be any of the two intersections of the circle with our ray. Let these intersections be C_1 and C_2, and let P be the midpoint of C_1C_2. Then, since $BP \perp C_1C_2$, we get that

$$\begin{aligned}
&[ABC_1] + [ABC_2] \\
&= BP \cdot AP \\
&= 14\sin(15°) \cdot 14\cos(15°) \\
&= 98(2\sin(15°)\cos(15°)) \\
&= 98\sin(30°) \\
&= 49.
\end{aligned}$$

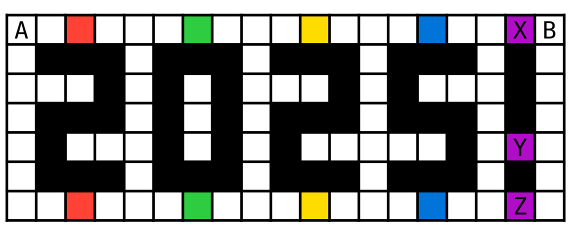
I8) Rupert starts on square A and wants to reach square B. He can repeatedly move to any white square that shares a side with his current square, but he cannot revisit any square he has already moved to. Find the number of paths Rupert can take to end up at square B.



Proposed by Zhenghua Xie.

Answer: 56

Solution:



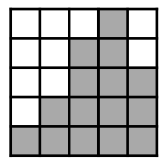
Consider each column of colored cells in the figure above. Any path will have to pass through exactly one of the red cells, one of the green cells, one of the yellow cells, and one of the blue cells. With the purple cells, the path either passes through exactly one of them (total of $2^4 \times 3$ possibilities) or enters through the bottom blue cell and passes through Z, then Y, then X (total of 2^3 possibilities). Our final answer is

$$2^4 \times 3 + 2^3 = 56.$$

I9) An n-tower is constructed by shading cells in an $n \times n$ grid according to the following rules:

- The kth row from the top has k consecutive shaded cells.
- Each shaded cell not in the bottom row is directly above another shaded cell.

An example of a 5-tower is shown below.



Let A be the number of 25-towers such that the middle cell of the 1st row from the top and the leftmost cell of the 21st row from the top are shaded. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

Proposed by Zhenghua Xie.

Answer: 59 $\left(\binom{20}{8} = 2 \times 3 \times 5 \times 13 \times 17 \times 19\right)$

Solution: Consider constructing the tower from the top down, going row by row. At each row, after shading in all squares present in the row above us, we can choose to extend the consecutive shaded cells either one cell to the left or one cell to the right.

Call a row a leftie if we extended to the left and call it a rightie if we extended one cell to the right. The middle cell of the first row being shaded tells us that, of the 24 other rows, 12 are lefties and 12 are righties. The leftmost cell of the 21st row being shaded tells us that the last 4 rows are all righties. Now, we need to choose 12 of the 20 rows in the middle to be lefties, so our answer is

$$\binom{20}{8} = 2 \times 3 \times 5 \times 13 \times 17 \times 19.$$

I10) Let ℓ be a non-vertical line that does not intersect the curve $y = \frac{x^4 - x^3}{x - 1}$. Let A be the smallest real number which is larger than all possible slopes of such a line ℓ. A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ryan Wu.

Answer: 304 $\left(\frac{3}{4}\right)$

Solution: The graph of $y = \frac{x^4 - x^3}{x - 1}$ is just $y = x^3$ but with a hole at $(1, 1)$. We need ℓ to pass through this hole. Let the equation of ℓ be $y = m(x - 1) + 1$. ℓ should intersect $y = x^3$ twice: once at $(1, 1)$ and another at a point of tangency. Let's compute the x-coordinate of all intersections between ℓ and $y = x^3$.

Substitute and rearrange to get

$$x^3 - mx + (m - 1) = 0.$$

We know $x = 1$ is a solution, so factoring out $(x - 1)$ from the left hand side gives

$$x^2 + x + (1 - m) = 0.$$

We want this quadratic to have exactly one solution, so we set its discriminant to 0 and solve to find that the maximum value of the slope is $\frac{3}{4}$.

I11) Let $\triangle ABC$ be a triangle with $BC = 32$ and circumradius 18. Let O be the circumcenter of $\triangle ABC$, and let D be the foot of the altitude from A to BC. Suppose that the area of $\triangle ABC$ is 3 times the area of $\triangle OBC$. Find OD^2.

Proposed by Rohan Mallick.

Answer: 120

Solution: Since $BC = 32$, $BM = 16$, and since $R = 18$, $OM = \sqrt{68}$. By area ratios, $AD = 3\sqrt{68}$. Since $AO = 18$, by the Pythagorean Theorem, $DM = \sqrt{18^2 - 4(68)} = \sqrt{52}$. Thus, by the Pythagorean Theorem, $OD^2 = 52 + 68 = 120$.

I12) Let $a_1, a_2, ..., a_8$ be a permutation of $2, 3, 3, 5, 6, 6, 7, 8$. a_i is considered a *prefix maximum* if either $i = 1$ or $a_i > \max(a_1, a_2, ..., a_{i-1})$. Define the *conspiratorial value* of a permutation to be the product of its prefix maximums. Let A be the sum of the conspiratorial values over all $\frac{8!}{2!2!}$ permutations. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

Proposed by Ryan Bai.

Answer: 48 $(2^6 \cdot 3^6 \cdot 7 \cdot 11)$

Solution: Let's instead compute the expected conspiratorial value of a random permutation. For $x \in \{2, 3, 5, 6, 7, 8\}$, let B_x be a random variable with value x if x is a prefix maximum and 1 otherwise. Note that for some $i < j$, B_i and B_j are independent, since whether j is a prefix maximum has nothing to do with whether i is a prefix maximum (j only cares about elements larger than j and disregards elements smaller than it).

We wish to find

$$\mathbb{E}[B_2 \cdot B_3 \cdot B_5 \cdot B_6 \cdot B_7 \cdot B_8],$$

which, because of the independence of B_i, is equal to

$$\mathbb{E}[B_2] \cdot \mathbb{E}[B_3] \cdot \mathbb{E}[B_5] \cdot \mathbb{E}[B_6] \cdot \mathbb{E}[B_7] \cdot \mathbb{E}[B_8].$$

Now, to actually compute, for example, B_6, we just need to compute the probability that, when we permute $6, 6, 7, 8$, the first element is a 6. Computing all $\mathbb{E}[B_i]$, multiplying them together, and then multiplying by $\frac{8!}{4}$ yields our final answer.

I13) Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 7$, $CA = 5$, and $\angle BAC = 120°$. Let P be a point inside $\triangle ABC$ such that $AP = 1$, and $\angle BPC = 150°$. Let the tangent line to the cirumcircle of $\triangle BPC$ at P intersect line BC at X, and let line AP intersect line BC at Y. The length XY can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Sepehr Golsefidy.

Answer: 4908 $\left(\frac{49}{8}\right)$

Solution: Essentially, note that if angle BPC is 150 degrees, we construct the point F such that BFC is equilateral. Note that since $FC = FB$ and angle BFC is 60 degrees, F is the center of (B P C). Now, $FP = 7$, but by Ptolemy's theorem, $AF = 8$, so P lies on AF. By Incenter-Excenter Lemma, P lies on the angle bisector of $\angle BAC$. Call the foot of this angle bisector to $BC = D$. Then, $AD = \frac{15}{8}$. If $XX' = x$, then by PoP, $\left(x - \frac{21}{8}\right)\left(x + \frac{35}{8}\right) = x^2 - \left(\frac{7}{8}\right)^2$, which gives $x = \frac{49}{8}$. We're done.

I14) Let x be the unique positive real number satisfying

$$x^{x^{x^{27}+27}} = 81^{3^{13}}.$$

Find the smallest integer n such that $x^n \geq 3^{64}$.

Proposed by Aidan Bai.

Answer: 432

Solution: We can manipulate the equation as follows.

$$\begin{aligned}
x^{x^{27}\cdot\left(x^{x^{27}}\right)} &= 3^{12\cdot 3^{12}} \\
\left(x^{x^{27}}\right)^{x^{x^{27}}} &= \left(3^{12}\right)^{3^{12}} \\
x^{x^{27}} &= 3^{12} \\
\left(x^{x^{27}}\right)^{27} &= \left(3^{12}\right)^{27} \\
x^{27\cdot x^{27}} &= 3^{4\cdot 81} \\
\left(x^{27}\right)^{x^{27}} &= 81^{81} \\
x^{27} &= 81 = 3^4 \\
x &= 3^{\frac{4}{27}}
\end{aligned}$$

Extracting our answer in the desired format yields a final answer of 432.

I15) Let

$$X = \sum_{i=1}^{2025} i^{2025}.$$

Let m be the largest integer such that 3^m divides X, and let n be the largest integer such that 5^n divides X. Find $m \times n$.

Proposed by Larry Wu.

Answer: 55 $(n = 11, m = 5)$

Solution: Let $\nu_p(x)$ for prime p denote the largest integer k for which $p^k \mid x$. We will first compute $\nu_3(X)$.

We will group i and $2025 - i$ and compute

$$\nu_3(X) = \nu_3(2X) = \nu_3\left(\sum_{i=0}^{2025} i^{2025} + (2025 - i)^{2025}\right).$$

Note that the terms where $3 \mid i$ have a ν_3 of at least 2025 and likely won't matter. The other terms have a ν_3 of 8 by the Lifting the Exponent Lemma.

From here, we conjecture that, since there are $\frac{2}{3} \times 2025 = 1350$ of these pairs, $m = 8 + \nu_3(1350) = 11$. To check this, we evaluate the sum modulo 3^{12} by expanding each term using the Binomial Theorem.

$$\begin{aligned} &\sum_{i=0}^{2025} i^{2025} + (2025 - i)^{2025} \\ \equiv &\sum_{i=0}^{2025} \binom{2025}{1} \cdot 2025 \cdot i^{2024} + \binom{2025}{2} \cdot 2025^2 \cdot i^{2023} + \cdots \end{aligned}$$

Note that the subsequent terms are all 0 modulo 3^{12}, so our sum becomes

$$\sum_{i=0}^{2025} 2025^2 \cdot i^{2024}.$$

Since $\nu_3(2025^2) = 8$, it suffices to evaluate $\sum_{i=0}^{2025} i^{2024}$ modulo 3^4.

Now let's group the terms by residue classes mod 3. We can ignore 0 mod 3. For 1 mod 3, we have

$$\sum_{k=0}^{674} (3k+1)^{2024} \equiv 675 + 3^1 \binom{2024}{1} \left(\sum_{k=0}^{674} k^1\right) + 3^2 \binom{2024}{2} \left(\sum_{k=0}^{674} k^2\right) + \cdots \pmod{3^4}.$$

Note that for $i \geq 1$, $675 \mid \sum_{k=0}^{674} k^i$, so all but the first term is 0 mod 3^4 and we have

$$\sum_{k=0}^{674} (3k+1)^{2024} \equiv 675 \pmod{3^4}.$$

We can do something similar for residue classes 2 mod 3, and we get

$$\sum_{k=0}^{674}(3k+2)^{2024} \equiv 675 \cdot 2^{2024} \equiv 675 \pmod{3^4},$$

where the last congruency holds because $2^{2024} \equiv 1 \pmod 3$ and $\nu_3(675) = 3$.

Thus,

$$\sum_{i=0}^{2025} i^{2024} \equiv 2 \cdot 675 \pmod{3^4},$$

so we have $\nu_3(X) = 8 + 3 = 11$.

A similar argument for ν_5 gives $\nu_5(X) = 5$. Our final answer is $\nu_3(X) \cdot \nu_5(X) = 11 \times 5 = 55$.

CCA Math Bonanza

March 8, 2025

Division I Team Round Solutions

T1) Larry is making a burger. He starts with a bun and can add any combination of lettuce, cheese, tomato, onion, and meat, including the option to leave the burger plain with just the bun. However, he doesn't want to make a burger that has meat but no cheese. The order in which he adds his ingredients doesn't matter. Find the number of different burgers Larry can create.

Proposed by Alex Backues.

Answer: 24

Solution: The burger either has:
- no meat and no cheese
- no meat and cheese
- meat and cheese

for a total of 3 combinations of meat and cheese. We can then multiply by the number of ways to determine if we include lettuce, tomato, and onion for a final answer of $3 \times 2^3 = 24$.

T2) The value

$$\frac{25!^2 - 24!^2}{25!^2 + 24!^2}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

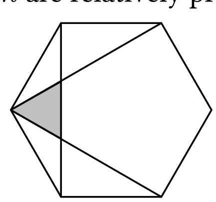
Proposed by Jason Yang.

Answer: 31513 $\left(\frac{312}{313}\right)$

Solution:

$$\frac{25!^2 - 24!^2}{25!^2 + 24!^2} = \frac{(24!)^2(25^2 - 1^2)}{(24!)^2(25^2 + 1^2)} = \frac{624}{626} = \frac{312}{313}$$

T3) The regular hexagon shown in the figure below has side length 1. The area of the shaded region can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.



Proposed by Jason Yang.

Answer: 148 $\left(\sqrt{\frac{1}{48}}\right)$

Solution: The shaded region is an equilateral triangle with a side length that is $\frac{1}{3}$ the length of the line segment across the hexagon. In fact, its side length is $\frac{\sqrt{3}}{3}$. The area of the equilateral triangle is then

$$\frac{\sqrt{3}}{4} * \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\sqrt{3}}{12} = \sqrt{\frac{1}{48}}$$

T4) The maximum possible value of $\frac{t}{3^t} - \left(\frac{t}{3^{t-1}}\right)^2$ over all real numbers t can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Lawrence Liu.

Answer: 136 $\left(\frac{1}{36}\right)$

Solution: Let $f(t) = \frac{t}{3^t}$. Substitute $x = \frac{t}{3^t}$, so we now wish to maximize

$$x - (3x)^2 = x - 9x^2.$$

This quadratic is maximized at $x = \frac{1}{18}$ yielding a value of $\frac{1}{36}$. Note that, by continuity, there exists some value of t for which $f(t) = \frac{1}{18}$, since $f(0) = 0$ and $f(1) = \frac{1}{3}$.

T5) Find the number of positive integers n such that $n \leq 750$ and $\gcd(1500, n)$ is a prime number.

Proposed by Andrew Jahng.

Answer: 240

Solution: If $\gcd(1500, n)$ is a prime number, then $\gcd(1500, 1500 - n) = \gcd(1500, n)$ is also a prime number. Furthermore, $n = 750$ doesn't work. Thus, it suffices to count the number of $n \leq 1500$ for which $\gcd(1500, n)$ is prime and divide by 2.

Let $\phi(n)$, Euler's totient function, be the number of positive integer $x \leq n$ for which $\gcd(n, x) = 1$. Let's count the number of $n \leq 1500$ for which $\gcd(1500, n) = 3$. Let $n = 3k$, with $k \leq 500$. We want $\gcd(1500, 3k) = 3$, so $\gcd(500, k) = 1$. The number of such k is given by $\phi(500)$. Repeating this for the other prime factors of 1500 gives an answer of

$$\phi\left(\frac{1500}{2}\right) + \phi\left(\frac{1500}{3}\right) + \phi\left(\frac{1500}{5}\right) = 200 + 200 + 80 = 480.$$

Dividing by 2 gives a final answer of 240.

T6) Three points A, B, and C are chosen uniformly at random along the circumference of a circle of radius 1. The probability that all angles in $\triangle ABC$ are less than 75° can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Zhenghua Xie.

Answer: 116 $\left(\frac{1}{16}\right)$

Solution: Note that the condition is equivalent to all three arcs formed by A, B, and C being less than 150°.

Let O be the center of the circle, and let $b = \angle AOB$ and $c = \angle AOC$ (b and c might be larger than 180°). Without loss of generality, assume $0° \leq b \leq 180°$. Now, we need $c \geq 180°$. In fact, the range of possible values of c is $\max(0, 300 - (360 - b)) = \max(0, b - 60)$, as long as $b < 150°$. We can now interpret this as a geometric probability, and our answer is

$$\frac{90 \times 90 \times \frac{1}{2}}{180 \times 360} = \frac{1}{16}.$$

T7) Let a, b, and c be real numbers that satisfy the following system of equations:

$$(a+1)(b+1)(c+1) - abc = 4$$
$$(a-1)(b-1)(c-1) + ab + bc + ca = 4$$
$$a + b + c + \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = 0$$

The sum of all possible values of $a^2 + b^2 + c^2$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Rohan Mallick.

Answer: 9304 $\left(\frac{93}{4}\right)$

Solution: Since all three equations are symmetric in a, b, and c, we let

$$s = a + b + c$$
$$q = ab + bc + ca$$
$$p = abc.$$

Rewriting in terms of s, q, and p, we get

$$s + q = 3$$
$$s + p = 5$$
$$q^2 = sp,$$

and the answer will be $s^2 - 2q$. Using the first 2 equations, we can write the last equation in terms of s to get

$$(3 - s)^2 = s(5 - s)$$
$$s^2 - 6s + 9 = 5s - s^2$$
$$2s^2 - 11s + 9 = 0$$
$$(s - 1)(2s - 9) = 0.$$

Thus, $s = 1$ or $s = \frac{9}{2}$. In the $s = 1$ case, we get $q = 2$, so $s^2 - 2q = -3$. In the $s = \frac{9}{2}$ case, we get that $q = -\frac{3}{2}$, so $s^2 - 2q = \frac{93}{4}$. Since a, b, c are real, -3 is extraneous. Thus, the answer is $\frac{93}{4}$.

Note: unfortunately, the solution set in the $\frac{93}{4}$ case also contains complex numbers, so the correct answer to this problem should be 0.

T8) Let $\triangle ABC$ be a triangle with $BC = 12$, $\angle BAC = 60°$, and $\angle ABC = 75°$. Let O, H, and I be the circumcenter, orthocenter, and incenter of $\triangle ABC$, respectively. If O' is the reflection of O over BC, then the area of $O'HIO$ can be expressed as $\sqrt{v} - \sqrt{w}$, where v and w are positive integers. Find $100v + w$.

Proposed by Rohan Mallick.

Answer: 86688 $\left(\sqrt{864} - \sqrt{288}\right)$

Solution: Note that since $\angle BAC = 60°$, $\angle BHC = \angle BIC = \angle BOC = 120°$, so B, H, I, O, and C all lie on a circle. Moreover, we get that triangles BOO' and COO' are equilateral, so O' is the circumcenter of $BHIOC$. Also, the circumradius of ABC is $\frac{12}{2\sin(60°)} = \frac{6}{\frac{\sqrt{3}}{2}} = 4\sqrt{3}$, so $AH^2 + BC^2 = \left(4\sqrt{3}\right)^2$. Solving for AH, we get $AH = 4\sqrt{3}$, and since $AH = AO = OO' = HO'$, $AHO'O$ is a rhombus. Then,

$$\angle HAI = \angle BAI - \angle BAH = \frac{\angle BAC}{2} - (90° - \angle ABC) = \frac{60°}{2} + (90° - 75°) = 30° - 15° = 15°.$$

Since H and O are isogonal conjugates, I lies on the angle bisector of $\angle HAO$, which means I lies on AO'. Thus, $\angle HO'I = \angle OO'I = 15°$, so

$$[O'HIO] = [O'OI] + [O'HI] = 2\left(\frac{1}{2}\left(4\sqrt{3}\right)^2 \sin(15°)\right)$$
$$= 48\sin(15°) = 48\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right) = 12\sqrt{6} - 12\sqrt{2} = \sqrt{864} - \sqrt{288}.$$

T9) Call a function $f(x)$ which takes in rational numbers as inputs and outputs real numbers *awesome* if, for any rational number x, $f(x) = f(2x) + f\left(\frac{x}{2}\right)$, and for any odd integer k, $f(k) = k$. Let S be a subset of $\{1, 2, \ldots, 100\}$ such that $\sum_{n \in S} f(n)$ has exactly one possible value across all *awesome* functions f. Find the maximum possible size of S.

Proposed by Noel Zhang.

Answer: 67

Solution: Let p be an odd number. Let $a = f\left(\frac{p}{2}\right)$, and let $b = p - a$. Now, we have

$$\begin{aligned} f\left(\frac{p}{2}\right) &= a \\ f(p) &= a+b \\ f(2p) &= b \\ f(4p) &= -a \\ f(8p) &= -a-b \\ f(16p) &= -b \\ f(32p) &= a \\ f(64p) &= a+b. \end{aligned}$$

Let $S_p = S \cap \{p, 2p, 4p, 8p, ...\}$. We can deal with $\sum_{n \in S_p} f(n)$ separately for each p. That sum is constant over all awesome functions only if it strictly a function of p (since otherwise we can change a and get a different sum).

We proceed with casework, calculating the maximum possible size of S_p for various values of p.

- $p \le \frac{100}{64}$: $f(p) + f(2p) + ... + f(64p) = p$
- $\frac{100}{64} < p \le \frac{100}{32}$: $f(p) + f(2p) + ... + f(32p) = 0$
- $\frac{100}{32} < p \le \frac{100}{16}$: either of the following work
  - $f(p) + f(8p) + f(4p) + f(16p) = -p$
  - $f(p) + f(8p) + f(2p) + f(16p) = 0$
- $\frac{100}{16} < p \le \frac{100}{8}$: $f(p) + f(8p) = 0$
- $\frac{100}{8} < p \le 100$: $f(p) = p$

Summing over all cases, we get $1 \times 7 + 1 \times 6 + 1 \times 4 + 3 \times 2 + 44 \times 1 = 67$.

T10) Let $\triangle ABC$ be a triangle with $BC = 120$ and $\angle BAC = 60°$. Let M be the midpoint of BC. It is given that $AM = 90$. Let Ω be the circumcircle of $\triangle ABC$, and let D be the midpoint of major arc BC. Let line AM intersect Ω at E, line DE intersect BC at F, and line AF intersect Ω at G. Let $K \neq G$ be the intersection of Ω with the circumcircle of MFG. Find AK.

Proposed by Sepehr Golsefidy.

Answer: 100

Solution: Symmedian Part: Extend KM to meet Ω at A'. Note that by angle arc formulas, $\angle AMF = \frac{\widehat{A'C}+\widehat{KB}}{2}$. Note that since K, M, F, G are cyclic, $\angle KGF = \angle AMF$. By arc angles, $\angle KGF = \angle KGA = \frac{\widehat{KA}}{2} = \frac{\widehat{KB}}{2} + \frac{\widehat{AB}}{2}$. Since these are equal, $\widehat{AB} = \widehat{A'C}$, which implies that the reflection of K over the perpendicular bisector of BC is E.

Computation: Notice that since $\widehat{BK} = \widehat{CE}$, $\angle BAK = \angle CAM$. Additionally, since $\angle MCA = \angle BCA = \angle BKA$, $\triangle CAM \sim \triangle KAB$. By similar triangles, $AK = \frac{(AB)(AC)}{AM}$. By median length formula, $180 = \sqrt{2(AB)^2 + 2(AC)^2 - 120^2}$, so $AB^2 + AC^2 = 23400$. By LoC, $AB^2 + AC^2 - (AB)(AC) = 14400$. Subtracting, we find $(AB)(AC) = 9000$. Thus, $AK = 100$.

CCA Math Bonanza

March 8, 2025

Division I Lightning Round Solutions

Set 1

Each problem in this set is worth 10 points.

L1.1) Aidan, Bryan, Charles, and Dylan walk into a movie theater and randomly sit down in a row of 4 seats, with all seating arrangements being equally likely. Unfortunately, Bryan hates both Aidan and Charles. The probability Bryan does not sit next to Aidan or Charles can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Tiger Han.

Answer: $106\left(\frac{1}{6}\right)$

Solution: Since the order at which they take a seat doesn't matter, we can assume Bryan sits down first and Dylan sits down second. If Bryan sits in one of the two center seats, he is guaranteed to sit next to one of Aidan and Charles. Thus, Bryan must sit in one of the corner seats (occurs with probability $\frac{1}{2}$), and then Dylan must sit next to Bryan (occurs with probability $\frac{1}{3}$). Multiplying gives an answer of $\frac{1}{6}$.

L1.2) Rupert the Raven and Freddy the Falcon are fighting over the last banana. They will each roll a fair n-sided die ($n > 4$), whose faces are labeled with the integers from 1 to n. If the larger of the two rolls is at most 4, Freddy wins. Otherwise, Rupert wins. Find the value of n such that Rupert is 15 times as likely to win as Freddy.

Proposed by Jason Yang.

Answer: 16

Solution: The probability that Freddy wins is $\frac{4^2}{n^2}$, and the probability that Rupert wins is $1 - \frac{4^2}{n^2}$. Thus, we have

$$15 \times \frac{4^2}{n^2} = 1 - \frac{4^2}{n^2},$$
$$16 \times \frac{4^2}{n^2} = 1,$$
$$16 \times 16 = n^2,$$

so $n = 16$.

L1.3) Find the value of

$$(2025 \times 2022 - 2024 \times 2023) + (2021 \times 2018 - 2020 \times 2019) + \cdots + (5 \times 2 - 4 \times 3) + 1.$$

Proposed by Zhenghua Xie.

Answer: -1011

Solution: We can rewrite each group of 4 as

$$(x+3)x-(x+2)(x+1)=-2.$$

There are $\left\lfloor \frac{2025}{4} \right\rfloor = 506$ such groups and an extra 1 at the end, so our answer is

$$-2\times 506+1=-1011.$$

L1.4) Let $ABCD$ be a rectangle. Let $CEFG$ be a square such that E lies on segment AB and F lies on segment CD. Given that $BE = DF$ and $DF + EF = 3$, the area of $ABCD$ can be expressed as $m - n\sqrt{2}$, where m and n are positive integers. Find $100m + n$.

Proposed by Aditya Bisain.

Answer: 8154 $\left(81-54\sqrt{2}\right)$

Solution: Note that CF is the diagonal of square $CEFG$, so $\triangle CEF$ is a 45-45-90 triangle. Let Y be the foot of the altitude from E to BC. We know $YC = YF = x$ and $FE = 3 - x$, so

$$3-x=\sqrt{2}x,$$

which yields $x = \frac{3}{1+\sqrt{2}}$. Now, $BC = x$ and $AB = 3x$, so our final answer is

$$3x^2 = 81-54\sqrt{2}.$$

Set 2

Each problem in this set is worth 12 points.

L2.1) Byan and Rai each pick a positive integer less than 100. It turns out that Byan's number is divisible by 5 and is greater than Rai's number. In addition, Rai's number divides Byan's number. Find the sum of all possible numbers Rai could've picked.

Proposed by Larry Wu.

Answer: 385

Solution: If Rai's number is divisible by 5, then Rai's number can be $5, 10, 15, \ldots, 45$ and Byan's number could be 2 times Rai's number. Otherwise, Rai's number must be less than or equal to 19 and Byan's number could be 5 times Rai's number.

Our answer is then

$$(1 + 2 + \cdots + 19) + (20 + 25 + \cdots + 45) = 190 + 190 = 380.$$

L2.2) Rupert the Raven has 4 independent, weighted coins, whose probabilities of landing heads on a flip are 10%, 40%, 55%, and p, where $0 \leq p \leq 1$. If he flips all 4 coins simultanenously, the probability he lands an odd number of heads is exactly 50%. The sum of all possible values of p can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ryan Bai.

Answer: 102 $\left(p = \frac{1}{2} \text{ only}\right)$

Solution: Imagine we flip the 3 coins first, and then the last one afterwards. Say the first 3 coins lands an odd number of heads with probability a. Then, we need

$$\begin{aligned} a(1-p) + p(1-a) &= 0.5 \\ -0.5 + a + p - 2ap &= 0, \end{aligned}$$

which factors as

$$(2a - 1)(2p - 1) = 0,$$

so one of a or p must be 0.5. Furthermore, applying the above argument and inducting shows that after flipping n weighted coins, the probability of an odd number heads is 0.5 only if at least one of the n coins is fair. Thus, $p = \frac{1}{2}$ is our only solution.

L2.3) A right circular cone containing an infinite series of spheres $S_0, S_1, S_2, \ldots$ has a base of radius 3 and a height of 4. Let S_0 be the largest sphere that can be placed entirely within the cone, and for $n > 0$, let S_n be the largest sphere that can be placed entirely within the space between the tip of the cone and S_{n-1}. The total volume of all the spheres can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Zhenghua Xie.

Answer: 3207 $\left(\frac{32}{7}\pi\right)$

Solution: Instead of calculating the radii in 3D, we can consider a cross section of the cone by slicing directly from the tip downwards. This will result in a triangle with circles since the spheres are tangent to the surfaces of the cone. Shifting perspective from 3D to 2D, this sum becomes an infinite sequence of incircles.

Our isosceles triangle has a height of length 4, base of length 6, and legs of length 5. The inradius of this triangle is given by $r = \frac{A}{s} = \frac{12}{8} = \frac{3}{2}$. Now, drawing a line tangent to the top of the incircle and parallel to the base of our isosceles triangle reveals a similar triangle. The smaller triangle has a height of $4 - 3 = 1$, so its inradius is $\frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$. Thus, the radii of the spheres form the following infinite sequence: $\frac{3}{2}, \frac{3}{8}, \frac{3}{32}, \frac{3}{128}, \ldots$

To finish, we compute the sum of the infinite geometric series

$$\frac{4\pi}{3}\left[\left(\frac{3}{2}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{3}{32}\right)^3 + \cdots\right],$$

which comes out to $\frac{32\pi}{7}$.

L2.4) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers such that

$$a_1 = a_2 = \cdots = a_{100} = 1,$$

and for all $n > 100$,

$$a_n = a_{n-100} + a_{n-99} + \cdots + a_{n-1}.$$

Find the second smallest $n > 1$ such that $a_n \neq 2a_{n-1} - 1$.

Proposed by Rohan Mallick.

Answer: 202

Solution: Note that $n > 100$. Thus, the smallest value of n that works is $n = 101$, because

$$a_{101} = a_{100} + a_{99} + \ldots + a_1 = 1 + 1 + \ldots + 1 = 100 \neq 1 = 2 \cdot 1 - 1 = 2a_{100} - 1.$$

Now, note that for $n > 101$,

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} + \cdots + a_{n-100} \\ &= a_{n-1} + a_{n-2} + \cdots + a_{n-100} + a_{n-101} - a_{n-101} \\ &= a_{n-1} + a_{n-1} - a_{n-101} \\ &= 2a_{n-1} - a_{n-101}. \end{aligned}$$

Thus, the condition $a_n \neq 2a_{n-1} - 1$ is equivalent to $a_{n-101} \neq 1$. The smallest such n is 202, since a_{101} is the first term that's not 1.

Set 3

Each problem in this set is worth 14 points.

L3.1) There exists a unique 5-digit palindrome of the form $\overline{abcba}$ that, when multiplied by $\frac{3}{4}$, becomes $\overline{a0cbc}$. Find $\overline{abcba}$.

Proposed by Larry Wu.

Answer: 27972

Solution: Expanding gives

$$-2499.25a + 747.5b - 26c = 0.$$

Note that a must be divisible by 2 for the left hand side to be an integer. In addition, we have

$$-2499.25a + 747.5b \geq 26c \geq 0,$$

so $b \geq 3.33a$. This forces $a = 2$ for b to be between 0 and 9. Now, either $b = 7$ or $b = 9$. b can't be 9, because in that case

$$747.5 \cdot 9 > 2500 \cdot 2 + 26 \cdot 10.$$

Thus, $b = 7$, and we can solve for c to get $a = 2, b = 7, c = 9$.

L3.2) The repeating decimal $0.\overline{ab}_k$ in base k can be written as $\frac{227}{306}$. For the minimum possible value of a, find $100a + b$.

Proposed by Larry Wu.

Answer: 2533 $(a = 25, b = 33)$

Solution: Note that minimizing a is equivalent to minimizing k. We have

$$\frac{ka + b}{k^2 - 1} = \frac{227}{306},$$

so $306 \mid k^2 - 1$. Note that $306 = 17 \times 18$, and 17 and 18 are consecutive numbers, so the smallest such k is 35. From here, we can compute a and b by expressing

$$(k^2 - 1) \times \frac{227}{306} = 908$$

in base 35, giving $a = 25, b = 33$.

L3.3) Rupert the Raven is playing a game. His score starts at 0. Every second, he flips a coin. If the coin lands heads, his score increases by 1. Otherwise, his score will reset to 0. Rupert repeats this for 100 coin flips. The expected number of times his score becomes 5 after a coin flip can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ryan Bai.

Answer: 9764 $\left(\frac{97}{64}\right)$

Solution: We use linearity of expectation. Let $f(x)$ be the expected number of times Rupert's score is at least x after a coin flip. Note that the expected number of times Rupert's score is 5 is $f(5) - f(6)$.

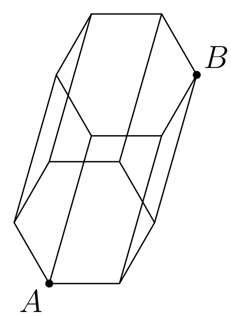
Now, for Rupert's score to be at least 5 after a coin flip, we just need the last 5 coin flips to all be heads. However, we need to ignore the first 4 coin flips. Thus, $f(5) = \frac{96}{32}$.

By a similar reasoning, $f(6) = \frac{95}{64}$, so our answer is

$$\frac{96}{32} - \frac{95}{64} = \frac{97}{64}.$$

Note: an alternative solution is to compute the expected number of times Rupert's score resets from some number at least 5. This is equivalent to computing the expected number of times Rupert flips, in this order, `HHHHHT`. We then need to account for Rupert ending the game with a score of at least 5.

L3.4) Shown below is a right regular hexagonal prism whose hexagonal base has side length 1. Larry the ant is currently standing at point A and wishes to reach point B. He notes that there are more than two distinct paths along the surface of the prism he can take to minimize the total distance he travels. The maximum possible height of the prism can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.



Proposed by Larry Wu.

Answer: 103 $\left(\sqrt{\frac{1}{3}}\right)$

Solution: By observation, it appears the only way there can be strictly greater than two distinct minimal paths from A to B is if the distance walking on the two rectangular faces is equal to the distance walking on one rectangular face and one hexagonal face. Note that these are also the only two types of paths that are ever optimal. Let the height of the prism (and rectangle) be h.

To compute the distances, we compute the distances between A and B when we "unfold" the faces of the prism into a net. Then, the distance from A to B along two rectangular faces is $\sqrt{h^2 + 4}$, and the distance along one rectangular face and one hexagonal face is

$$\sqrt{\left(h + \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2}.$$

Equating these and expanding, we get the equation

$$h^2 + h\sqrt{3} + 3 = h^2 + 4.$$

Simplifying, we find that $h\sqrt{3} = 1$, so $h = \sqrt{\frac{1}{3}}$.

Set 4

Each problem in this set is worth 16 points.

L4.1) Find the sum of $x + y$ over all pairs of positive integers x and y satisfying

$$x^2 + y^2 = 193(x - y).$$

Proposed by Katrina Liu.

Answer: 263 $((60, 35), (133, 35))$

Solution: Rewrite the equation as

$$(x + y)^2 + (193 - x + y)^2 = 193^2,$$

yielding a Pythagorean triple. We now apply the general formula for Pythagorean triples. Let $193 = k(u^2 + v^2)$. Since 193 is prime, we solve for $u^2 + v^2 = 193$, yielding $(u, v) = (12, 7)$. Thus, our Pythagorean triple is

$$(12^2 - 7^2, 2 \times 12 \times 7, 193) = (95, 168, 193).$$

When $x + y = 95$ and $193 - x + y = 168$, we get $x = 60$ and $y = 35$.

When $x + y = 168$ and $193 - x + y = 95$, we get $x = 133$ and $y = 35$.

L4.2) Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$, and let its incircle touch BC, CA, and AB at D, E, and F, respectively. Let I be the incenter of $\triangle ABC$. Let X be the intersection of lines EF and BC, and let Y be the intersection of lines XI and AD. $\frac{YE}{YF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Rohan Mallick.

Answer: 5245 $\left(\frac{52}{45}\right)$

Solution: Let γ be the incircle, let $D' \neq D$ be the second intersection of AD and γ, and let Z be the intersection of AD and EF. Then, $DED'F$ is harmonic, so XD must be tangent to γ. Thus, $DD' \perp XI$, and since $\angle AEI = \angle AFI = \angle AYI$, $AEYIF$ is cyclic. Then, since $\angle EAI = \angle FAI$, $IE = IF$, so by Circle Ratio Lemma, $\frac{YE}{YF} = \frac{XE}{XF} = \left(\frac{DE}{DF}\right)^2$.

Note that since $\triangle CDE$ is isosceles, we get $DE = 2CD \sin\left(\frac{\angle ACB}{2}\right)$. Using the Law of Cosines, we can find that $\cos(\angle ACB) = \frac{3}{5}$, so $\sin\left(\frac{\angle ACB}{2}\right) = \frac{1}{\sqrt{5}}$. Thus, $DE = 2\left(\frac{14+15-13}{2}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{16}{\sqrt{5}}$. Using a similar process, we can find that $DF = \frac{24}{\sqrt{13}}$, and plugging our values for DE and DF into the expression we had earlier, we get that $\frac{YE}{YF} = \frac{52}{45}$.

L4.3) Find the value of

$$2048 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \frac{1}{xy(x+2)(x+y+4)}.$$

Proposed by Rebecca Luo.

Answer: 576

Solution: Let

$$S = 2048 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \frac{1}{xy(x+2)(x+y+4)}.$$

Now, by swapping x and y in the sum and adding S to itself, we have

$$\begin{aligned} 2S &= 2048 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \left(\frac{1}{xy(x+2)(x+y+4)} + \frac{1}{yx(y+2)(y+x+4)} \right) \\ &= 2048 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \left(\frac{x+y+4}{xy(x+y+4)(x+2)(y+2)} \right) \\ &= 2048 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \left(\frac{1}{x(x+2)} \cdot \frac{1}{y(y+2)} \right) \\ &= 2048 \left(\sum_{x=1}^{\infty} \frac{1}{x(x+2)} \right) \left(\sum_{y=1}^{\infty} \frac{1}{y(y+2)} \right) \end{aligned}$$

We now evaluate

$$\sum_{x=1}^{\infty} \frac{1}{x(x+2)} = \frac{1}{2} \sum_{x=1}^{\infty} \frac{1}{x} - \frac{1}{x+2} = \frac{3}{4}.$$

Our final answer is then

$$1024 \times \left(\frac{3}{4} \right)^2 = 576.$$

L4.4) Let

$$A = \sum_{n=0}^{12} n^2 \binom{12}{n}^2.$$

Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

Proposed by Aidan Bai.

Answer: 79 $\left(12 \cdot 12 \cdot \binom{22}{11}\right) = 2^7 \times 3^3 \times 7^1 \times 13^1 \times 17^1 \times 19^1\right)$

Solution: We will establish a bijection. Suppose there are 12 boys and 12 girls that want to form two groups of 12 numbered 1 and 2 such that each group has a captain, group 1 has a male captain, and group 2 has a female captain. By iterating on the number of males in group 1, we get the summation above. However we can also find this by choosing a male captain in 12 ways, choosing a female captain in 12 ways, and splitting the remaining 22 into two groups of 11 for a total of

$$12 \cdot 12 \cdot \binom{22}{11} = 2^7 \times 3^3 \times 7^1 \times 13^1 \times 17^1 \times 19^1.$$

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

L5.1) The mex, or minimum excluded value, of a list of nonnegative integers is defined as the smallest nonnegative integer that does not appear in the list. For example, the mex of $0, 9, 2, 1, 1, 0$ is 3, because 0, 1, and 2 appear, but 3 does not.

Submit a nonnegative integer X. If M is the mex of all submissions to this problem, you will receive $\left\lfloor \frac{20}{(M-X)^2} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

Proposed by Larry Wu.

Answer: The mex of all submissions was 7.

Solution: The Nash equilibria of this game are:
- everyone submits 1 so the mex is 0
- everyone submits either 0 or 2, with at least one person submitting 0 so the mex is 1

In theory, it is always at least as optimal to submit 1 instead of 0. In practice, it turns out we had 4 teams across both divisions submit 0. All valid submissions we received are below. The modal submission of 6 received full points.

0	0	0	0	1	1	1	1	1
2	2	2	3	3	4	4	4	5
5	5	6	6	6	6	6	6	6
9	9	10	12	13	16	17	18	19
22	22	23	26	26	29	32	34	49

L5.2) Let N be the largest positive integer such that no substring of its decimal expansion is divisible by 193. Estimate the sum of the squares of the digits of N.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot \min\left(\frac{A}{E}, \frac{E}{A}\right)^{13} \right\rfloor$ points.

Hint: 193 is a prime. The repeating part of the decimal expansion of $\frac{1}{193}$ has period 192.

Proposed by Katrina Liu.

Answer: 13175

Solution: Consider the value of all prefixes of N padded with 0s mod 193. For example, if $N = 123$, then consider the set

$$\{0, 100, 120, 123\}.$$

The condition of the problem is equivalent to all elements of this set being distinct modulo 193. This is because any substring of N can be expressed as the difference of two prefixes divided by some power of 10. By pigeonhole, N should have at most 192 digits, and intuitively, we expect that it should (reasoning explained below).

Intuitively, we expect N to start with a bunch of 9s, then have some random digits at the end. Let's say the prefix of N has A 9s and the suffix of random digits has length B (where $A + B = 192$). We're basically trying to see how many 9s we can have N start with while still making it possible to pad in the rest of the digits. Since the order of 10 mod 193 is 192, as long as we don't have 192 9s, the prefix values of the prefix of 9s should not conflict with each other.

Let's estimate the probability that we can find a suffix of B random digits so that our set of prefix values is unique. When we pick such a suffix, let's assume that the prefix values in the suffix are all randomly distributed. The prefix values generated by the leading A 9s forces our suffix prefix values to be some permutation of B numbers. Thus, our probability is

$$f(B) = \frac{B!}{193^B}.$$

Our probability of finding some valid suffix across all 10^B suffixes is then

$$g(B) = 1 - (1 - f(B))^{10^B} \approx 10^B f(B) = \frac{B!}{19.3^B},$$

where we can make the approximation in the middle because $f(B)$ is close to 0.

Now, $g(B)$ is a cumulative probability distribution. It's also very fast growing, so it suffices to compute when $g(B) = 1$, so

$$B! = 19.3^B.$$

Stirling's approximation gives

$$\begin{aligned} B \ln B - B &= B \ln(19.3), \\ \ln\left(\frac{B}{e}\right) &= \ln(19.3), \\ B = 19.3e &\approx 19.3 \times 2.7 = 52.11. \end{aligned}$$

Now, we extract our answer. We assume the B digits of the suffix are random, so the expected value of the square of each of them is $\frac{95}{3}$. Our answer is then approximately

$$(192 - 52) \times 9^2 + 52 \times \frac{95}{3} = 12986.66.$$

The actual answer is:

```
999999999999999999999
999999999999999999999
999999999999999999999
999999999999999999999
999999999999999999999
999999999999999999999
999999999999999999999
61251582972783139996
```

12552435648859296487
133796998374

L5.3) The sequence of Lucas numbers is defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Estimate the number of integers between 1 and 2025, inclusive, that can be expressed as the sum of 3 (not necessarily distinct) Lucas numbers.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot 0.98^{|A-E|} \right\rfloor$ points.

Proposed by Zhenghua Xie.

Answer: 469

Solution: First, let's list out the Lucas numbers under 2025.

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207.$$

Observe that expressing a number as the sum of one or two Lucas numbers generally implies it can be written as the sum of three Lucas numbers (some of which are necessarily consecutive), unless the number is < 3. Note that $L_0 + L_2 = L_1 + L_3$. Also note that the "not necessarily distinct" condition is useless because $2L_n = L_{n+1} + L_{n-2}$, and if $n < 2$, then we can trivially see $2L_1 = L_0$ and $2L_0 = L_3$. Then, the number of numbers less than 2025 expressable as the sum of one or two Lucas numbers is about $14 + \binom{15}{2} - 1 = 119$.

Consider the case for three Lucas numbers. If two of them are consecutive, then this is reducable to the two Lucas numbers case. Then, we just have to consider the number of ways to take 3 non-consecutive Lucas numbers from the first 16 Lucas numbers, which there are $\binom{14}{3} = 364$ ways to do. Note that exactly one of these, $L_{15} + L_{13} + L_{11}$ is greater than 2025, so we will subtract one from the total. Again, note that $L_0 + L_2 = L_1 + L_3$, so another 12 should be subtracted from the total.

Summing it all up, we get approximately $119 + 364 - 1 - 12 = 469$ as our estimate, which happens to be exactly correct.

L5.4) Submit an integer between 1 and 240, inclusive. Your *index* is initially 0. Every positive integer divisor x of your submission will add $\frac{1}{n}$ points to your index, where n is the number of submissions to this problem (including yours) that are divisible by x.

If your index is S and the maximum index amongst all teams is M, you will receive $\left\lfloor 20 \cdot \left(\frac{S}{M}\right)^{1.5} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

Proposed by Rohan Mallick.

Answer: Here are all the valid submissions we received:
2 6 7 24 24 26 40 69 77 120 120 128 144 147 154 180 192 200 209 209 210 210 210 210 210 212 212 215 221 224 230 234 240 240 240 240 240 240 240 240 240 240

Solution: Here's a list of the best numbers to submit against the pool of submissions we received during the contest. Note that only the top two submissions would have beaten the previous best answer of $234 = 2 \times 3^2 \times 13$, which received an index of 5.63. Also of note is that the next three best submissions, $224 = 2^5 \times 7$ with an index of 4.79, $180 = 2^2 \times 3^2 \times 5$ with an index of 4.75, and $200 = 2^3 \times 5^2$ with an index of 4.38 are still better than some of the entries in this list.

Number	**Index**
$204 = 2^2 \times 3 \times 17$	5.73
$216 = 2^3 \times 3^3$	5.67
$228 = 2^2 \times 3 \times 19$	5.56
$198 = 2 \times 3^2 \times 11$	5.33
$220 = 2^2 \times 5 \times 11$	4.94
$132 = 2^2 \times 3 \times 11$	4.93
$162 = 2 \times 3^4$	4.63
$232 = 2^3 \times 29$	4.14
$222 = 2 \times 3 \times 37$	4.13
$186 = 2 \times 3 \times 31$	4.13

CCA Math Bonanza

March 8, 2025

Division I Tiebreaker Round Solutions

TB1) Let n and x be positive integers such that the sum of the digits of x is 2 and

$$9900\,9900\,9900\,9901 \times n = x.$$

Find the smallest such n.

Proposed by Alex Backues.

Answer: 101

Solution: Note that since the digits of x sum to 2, x is either of the form $2 \cdot 10^k$ or 100…000100… However, since x is divisible by 9900 9900 9900 9901, it can't be of the form $2 \cdot 10^k$.

x starts with a 1, so n must start with a 1 or 2, since all other digits will cause the leading digit of x to not be 1. Furthermore, x's units digit is either 0 or a 1, so n's units digit must also be either 0 or 1. Checking all numbers of this form reveals that 101 is the smallest solution.

TB2) Charlotte the cat lives in Cartesia, a city on the coordinate plane whose roads are the lines $x = a$ and $y = a$ for integers a. Charlotte is currently standing at the origin, and would like to walk to the highway at $y = 5$, and then to her home at $(6, 1)$. Let D be the length of the shortest possible path she could take home. Find the distinct number of paths of length D Charlotte can take.

Proposed by Rohan Mallick.

Answer: 5005

Solution: Note that for any point P on $y = 5$, the distance from P to $(6, 1)$ is the same as the distance from P to $(6, 9)$, because $(6, 1)$ and $(6, 9)$ are reflections across $y = 5$. Thus, we can instead consider journeys from $(0, 0)$ to $y = 5$ to $(6, 9)$. However, journeys with the shortest possible path from $0, 0$ to $y = 5$ to $(6, 9)$ are equivalent to journeys with the shortest possible path from $(0, 0)$ to $(6, 9)$, which means the answer is $\binom{15}{6} = 5005$.

TB3) A positive integer n is *x-central* if there exists integers a and $-2 \leq b \leq 4$ such that $n = ax + b$. Find the number of positive integers $k < 8390$ that satisfy exactly 2 of the following:

- k is 13-central
- k is 17-central
- k is 19-central

Proposed by Zhenghua Xie.

Answer: 2744

Solution: Compute $13 \cdot 17 \cdot 19 \cdot 2$ as 8398. Note that 8391 to 8398 do not work, as they satisfy exactly 0 or 3 of the conditions. Thus, it suffices to compute the number of positive integers $k \leq 8398$ which satisfy the above conditions.

Applying the Chinese Remainder Theorem, for each triple of (b_{13}, b_{17}, b_{19}) such that

$$0 \le b_{13} < 13,$$
$$0 \le b_{17} < 17,$$
$$0 \le b_{19} < 19,$$

there exists exactly two $0 < k \le 8398$ such that

$$k \equiv b_{13} \pmod{13},$$
$$k \equiv b_{17} \pmod{17},$$
$$k \equiv b_{19} \pmod{19}.$$

It suffices to count the number of valid (b_{13}, b_{17}, b_{19}). We can casework on whether we break the first, second, or last condition. Our final answer is

$$2 \cdot (6 \cdot 7 \cdot 7 + 7 \cdot 10 \cdot 7 + 7 \cdot 7 \cdot 12) = 2 \cdot 49 \cdot 28 = 2744.$$

TB4) Unit cube $ABCD - EFGH$ has square faces $ABCD$ and $EFGH$, with vertices A, B, C, D adjacent to vertices E, F, G, H, respectively. Two regular tetrahedrons with bases DBE and DBG are constructed. Let the apexes of the two tetrahedrons be P and Q, with both P and Q lying outside the cube. The length PQ can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Kevin Yang.

Answer: 5009 $\left(\sqrt{\frac{50}{9}}\right)$

Solution: Begin by taking one tetrahedron, say $BDHP$. let the center of square $ABCD$ be O, and extend OC to meet the drop down perpendicular from P at point M. PM is perpendicular to plane $EFGH$. We will solve for the lengths in triangle OPM. Draw cevian PC, and mark MC as length X, PM as length Y. Find PC by subtracting the height of tetrahedron $BDCH$ from $BDHP$ to get that $PC = \frac{\sqrt{3}}{3}$. Now since $OC = \frac{\sqrt{2}}{2}$ and $PO = \frac{\sqrt{6}}{2}$, we get that $\left(X + \frac{\sqrt{2}}{2}\right)^2 + Y^2 = \frac{3}{2}$. Using this equation, along with $X^2 + Y^2 = \frac{1}{3}$ gives that $X = \frac{\sqrt{2}}{3}$. Hence,the length $PQ = 2OM = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{3}\right) = \frac{5\sqrt{2}}{3}$