CCA Math Bonanza

March 8, 2025

Division I Lightning Round

Set 1

Each problem in this set is worth 10 points.

L1.1) Aidan, Bryan, Charles, and Dylan walk into a movie theater and randomly sit down in a row of 4 seats, with all seating arrangements being equally likely. Unfortunately, Bryan hates both Aidan and Charles. The probability Bryan does not sit next to Aidan or Charles can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L1.2) Rupert the Raven and Freddy the Falcon are fighting over the last banana. They will each roll a fair n-sided die ($n > 4$), whose faces are labeled with the integers from 1 to n. If the larger of the two rolls is at most 4, Freddy wins. Otherwise, Rupert wins. Find the value of n such that Rupert is 15 times as likely to win as Freddy.

L1.3) Find the value of

$$(2025 \times 2022 - 2024 \times 2023) + (2021 \times 2018 - 2020 \times 2019) + \cdots + (5 \times 2 - 4 \times 3) + 1.$$

L1.4) Let $ABCD$ be a rectangle. Let $CEFG$ be a square such that E lies on segment AB and F lies on segment CD. Given that $BE = DF$ and $DF + EF = 3$, the area of $ABCD$ can be expressed as $m - n\sqrt{2}$, where m and n are positive integers. Find $100m + n$.

Set 2

Each problem in this set is worth 12 points.

L2.1) Byan and Rai each pick a positive integer less than 100. It turns out that Byan's number is divisible by 5 and is greater than Rai's number. In addition, Rai's number divides Byan's number. Find the sum of all possible numbers Rai could've picked.

L2.2) Rupert the Raven has 4 independent, weighted coins, whose probabilities of landing heads on a flip are 10%, 40%, 55%, and p, where $0 \leq p \leq 1$. If he flips all 4 coins simultanenously, the probability he lands an odd number of heads is exactly 50%. The sum of all possible values of p can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L2.3) A right circular cone containing an infinite series of spheres $S_0, S_1, S_2, \ldots$ has a base of radius 3 and a height of 4. Let S_0 be the largest sphere that can be placed entirely within the cone, and for $n > 0$, let S_n be the largest sphere that can be placed entirely within the space between the tip of the cone and S_{n-1}. The total volume of all the spheres can be expressed as $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $100m + n$.

L2.4) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers such that

$$a_1 = a_2 = \cdots = a_{100} = 1,$$

and for all $n > 100$,

$$a_n = a_{n-100} + a_{n-99} + \cdots + a_{n-1}.$$

Find the second smallest $n > 1$ such that $a_n \neq 2a_{n-1} - 1$.

Set 3

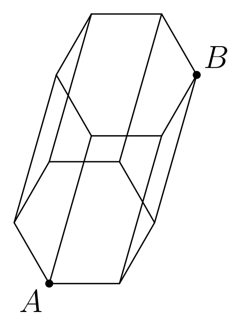
Each problem in this set is worth 14 points.

L3.1) There exists a unique 5-digit palindrome of the form $\overline{abcba}$ that, when multiplied by $\frac{3}{4}$, becomes $\overline{a0cbc}$. Find $\overline{abcba}$.

L3.2) The repeating decimal $0.\overline{ab}_k$ in base k can be written as $\frac{227}{306}$. For the minimum possible value of a, find $100a + b$.

L3.3) Rupert the Raven is playing a game. His score starts at 0. Every second, he flips a coin. If the coin lands heads, his score increases by 1. Otherwise, his score will reset to 0. Rupert repeats this for 100 coin flips. The expected number of times his score becomes 5 after a coin flip can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L3.4) Shown below is a right regular hexagonal prism whose hexagonal base has side length 1. Larry the ant is currently standing at point A and wishes to reach point B. He notes that there are more than two distinct paths along the surface of the prism he can take to minimize the total distance he travels. The maximum possible height of the prism can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $100m + n$.



Set 4

Each problem in this set is worth 16 points.

L4.1) Find the sum of $x + y$ over all pairs of positive integers x and y satisfying

$$x^2 + y^2 = 193(x - y).$$

L4.2) Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$, and let its incircle touch BC, CA, and AB at D, E, and F, respectively. Let I be the incenter of $\triangle ABC$. Let X be the intersection of lines EF and BC, and let Y be the intersection of lines XI and AD. $\frac{YE}{YF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

L4.3) Find the value of

$$2048 \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \frac{1}{xy(x+2)(x+y+4)}.$$

L4.4) Let

$$A = \sum_{n=0}^{12} n^2 \binom{12}{n}^2.$$

Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

Set 5

This is an estimation round. The scoring for each problem is specified in the problem statement. The maximum possible score for a problem is 20 points.

L5.1) The mex, or minimum excluded value, of a list of nonnegative integers is defined as the smallest nonnegative integer that does not appear in the list. For example, the mex of $0, 9, 2, 1, 1, 0$ is 3, because 0, 1, and 2 appear, but 3 does not.

Submit a nonnegative integer X. If M is the mex of all submissions to this problem, you will receive $\left\lfloor \frac{20}{(M-X)^2} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.

L5.2) Let N be the largest positive integer such that no substring of its decimal expansion is divisible by 193. Estimate the sum of the squares of the digits of N.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot \min\left(\frac{A}{E}, \frac{E}{A}\right)^{13} \right\rfloor$ points.

Hint: 193 is a prime. The repeating part of the decimal expansion of $\frac{1}{193}$ has period 192.

L5.3) The sequence of Lucas numbers is defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Estimate the number of integers between 1 and 2025, inclusive, that can be expressed as the sum of 3 (not necessarily distinct) Lucas numbers.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20 \cdot 0.98^{|A-E|} \right\rfloor$ points.

L5.4) Submit an integer between 1 and 240, inclusive. Your *index* is initially 0. Every positive integer divisor x of your submission will add $\frac{1}{n}$ points to your index, where n is the number of submissions to this problem (including yours) that are divisible by x.

If your index is S and the maximum index amongst all teams is M, you will receive $\left\lfloor 20 \cdot \left(\frac{S}{M}\right)^{1.5} \right\rfloor$ points.

Note: submissions from both divisions will be combined when calculating scores.