CCA Math Bonanza

March 8, 2025

Division I Individual Round

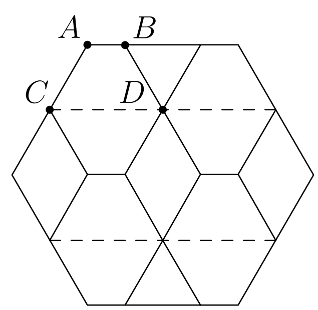
I1) Let $a \heartsuit b$ be the result of dividing a by b and keeping only the integer part. For example, $11 \heartsuit 3 = 3$ because $\frac{11}{3} = 3\,\frac{2}{3}$, and $29 \heartsuit 7 = 4$ because $\frac{29}{7} = 4\,\frac{1}{7}$. Find $(2025 \heartsuit 3) \heartsuit (2024 \heartsuit 7)$.

I2) Find $147 \times \left(67^2 - 64^2\right) + 3^2 \times 23 \times 147$.

I3) The sum of the coefficients (including the constant term) of the polynomial $\left(x^2 + ax - 4\right)^2$ is 36. Find the sum of the squares of all possible values of a.

I4) Let y be an integer, and let $x = 5^4 - 5^2 y^2$. Given that x is an even positive integer, find the largest possible number of positive divisors of x.

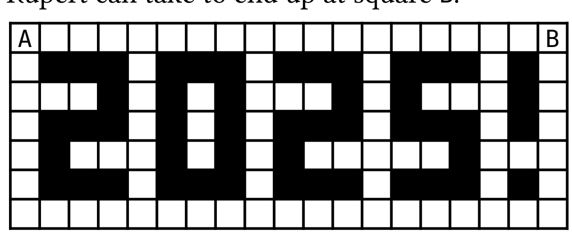
I5) The diagram below shows eight congruent isosceles trapezoids arranged inside a regular hexagon. $\frac{AB}{CD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.



I6) Byan picks a random integer between 0 and 3, inclusive. Then, Rai flips 4 fair coins. The probability that the amount of heads Rai lands is equal to Byan's integer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I7) Let $\triangle ABC$ be a triangle with $AB = 14$, $BC = 9$, and $\angle BAC = 15°$. Find the sum of all possible values of the area of $\triangle ABC$.

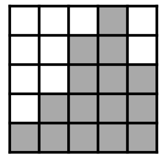
I8) Rupert starts on square A and wants to reach square B. He can repeatedly move to any white square that shares a side with his current square, but he cannot revisit any square he has already moved to. Find the number of paths Rupert can take to end up at square B.



I9) An n-tower is constructed by shading cells in an $n \times n$ grid according to the following rules:

- The kth row from the top has k consecutive shaded cells.
- Each shaded cell not in the bottom row is directly above another shaded cell.

An example of a 5-tower is shown below.



Let A be the number of 25-towers such that the middle cell of the 1st row from the top and the leftmost cell of the 21st row from the top are shaded. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

I10) Let ℓ be a non-vertical line that does not intersect the curve $y = \frac{x^4 - x^3}{x-1}$. Let A be the smallest real number which is larger than all possible slopes of such a line ℓ. A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I11) Let $\triangle ABC$ be a triangle with $BC = 32$ and circumradius 18. Let O be the circumcenter of $\triangle ABC$, and let D be the foot of the altitude from A to BC. Suppose that the area of $\triangle ABC$ is 3 times the area of $\triangle OBC$. Find OD^2.

I12) Let $a_1, a_2, \ldots, a_8$ be a permutation of $2, 3, 3, 5, 6, 6, 7, 8$. a_i is considered a *prefix maximum* if either $i = 1$ or $a_i > \max(a_1, a_2, \ldots, a_{i-1})$. Define the *conspiratorial value* of a permutation to be the product of its prefix maximums. Let A be the sum of the conspiratorial values over all $\frac{8!}{2!2!}$ permutations. Find the sum of the prime factors of A, counting multiplicity. For example, the sum of the prime factors of 12, counting multiplicity, is $2 + 2 + 3 = 7$.

I13) Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 7$, $CA = 5$, and $\angle BAC = 120°$. Let P be a point inside $\triangle ABC$ such that $AP = 1$, and $\angle BPC = 150°$. Let the tangent line to the cirumcircle of $\triangle BPC$ at P intersect line BC at X, and let line AP intersect line BC at Y. The length XY can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I14) Let x be the unique positive real number satisfying

$$x^{x^{x^{27}+27}} = 81^{3^{13}}.$$

Find the smallest integer n such that $x^n \geq 3^{64}$.

I15) Let

$$X = \sum_{i=1}^{2025} i^{2025}.$$

Let m be the largest integer such that 3^m divides X, and let n be the largest integer such that 5^n divides X. Find $m \times n$.